Undergraduate Research

On the Functional Equation $f(x+y)+f(x-y)=kf(x)f(y)$

By Keturah LaKena Moore and Hussain Elalaoui-Talibi

Biographical Sketch

Keturah LaKena Moore, is the daughter of Mr. and Mrs. William L. Moore of Huntsville, AL, and a 1999 graduate of J.O. Johnson High School in Huntsville, Alabama. Currently a senior at Tuskegee University, she will be receiving a B.A. in Mathematics in May 2005. Keturah's immediate plans include teaching mathematics at the high school level, while pursuing a masters degree. Her long term goal is to open a magnet school with a focus on Science and Mathematics. She is an active volunteer for *The Tuskegee Institute National Park Service,* and in 2004 she held the position of the first *Rock-the-Vote* coordinator in the state of Alabama. Keturah also received an honorary award from *The National Dean's List* in Spring of 2004. This paper was written under the direction and supervision of Dr. Hussain Talibi, her senior seminar advisor at Tuskegee University.

ABSTRACT: We find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ that satisfy the condition $f(x+y) + f(x-y) = kf(x) f(y)$ for all $(x, y) \in \mathbf{R}^2$, where k is a nonzero real number.

Introduction

Taking a careful look at the sum and difference formulas for the cosine function:

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y), \text{ and}$$

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin y,$$

we can add them and get the equation:

$$\cos(x+y) + \cos(x-y) = 2\cos(x)\cos(y).$$

Similarly, we can do the same thing with the hyperbolic cosine function to get:

$$\cosh(x+y) + \cosh(x-y) = 2\cosh(x)\cosh(y).$$

A natural question to ask is: "what continuous functions satisfy the general equation:

$$f(x+y) + f(x-y) = kf(x) f(y), \tag{1}$$

for all $(x, y) \in \mathbf{R}^2$, where k is a nonzero real number?"

We will show that other than some trivial functions, the cosine and hyperbolic cosine functions are indeed the only ones that satisfy that functional equation.

Main Results

Clearly $f(x) \equiv 0$ satisfies the equation. Consequently, we will assume throughout that $f(x)$ is not identically zero, that is, $f(z) \neq 0$ for some $z \in \mathbf{R}$.

LEMMA 1. *If f satisfies (1), then f is even, and $f(0) = \frac{2}{k}$.*

PROOF. By letting $x = y = 0$ in (1), we get $2f(0) = kf^2(0)$, so $f(0)(2 - kf(0)) = 0$, which shows that either $f(0) = 0$ or $f(0) = \frac{2}{k}$. Since, by hypothesis, $f(z) \neq 0$ for some $z \in \mathbf{R}$, we can choose x such that $f(x) \neq 0$ and let $y = 0$. Then (1) becomes

$2f(x) = kf(x)f(0)$, which implies that $f(0) \neq 0$, and hence, $f(0) = \frac{2}{k}$.

Next, if we let $y \in \mathbf{R}$ and let $x = 0$, then (1) becomes $f(y) + f(-y) = kf(y)f(0) = 2f(y)$. So $f(-y) = f(y)$, which shows that f is even. □

LEMMA 2. *If f is continuous and satisfies (1), then f is infinitely differentiable.*

PROOF. Since f is not identically zero and is continuous, there exists an interval $[a,b]$ such that $\int_a^b f(z)\,dz \neq 0$. So $\int_a^b f(x+y) + f(x-y)\,dy = kf(x)\int_a^b f(y)\,dy$, and it follows that

$$f(x) = M\left(\int_a^b f(x+y)\,dy + \int_a^b f(x-y)\,dy\right)$$

where $M = \frac{1}{k\int_a^b f(y)dy}$ is a constant. If we use the substitution, $u = x+y$, we get $\int_a^b f(x+y)\,dy = \int_{x+a}^{x+b} f(u)\,du = \int_{-b-x}^{-a-x} f(u)\,du$, since f is even. Similarly, using the substitution, $u = x - y$, we have $\int_a^b f(x-y)\,dy = \int_{a-x}^{b-x} f(u)\,du$. Now we can write:

$$\begin{aligned} f(x) &= M\left(\int_{a-x}^{b-x} f(u)\,du + \int_{-b-x}^{-a-x} f(u)\,du\right) \\ &= M\left(\int_{a-x}^{0} f(u)\,du + \int_0^{b-x} f(u)\,du \right. \\ &\qquad \left. + \int_{-b-x}^{0} f(u)\,du + \int_0^{-a-x} f(u)\,du\right) \\ &= M\left(-\int_0^{a-x} f(u)\,du + \int_0^{b-x} f(u)\,du \right. \\ &\qquad \left. - \int_0^{-b-x} f(u)\,du + \int_0^{-a-x} f(u)\,du\right) \end{aligned}$$

So $f'(x)$ exists, is continuous, and is equal to $f(-b-x) - f(-a-x)$ by the Fundamental Theorem of Calculus. Since $f'(x)$ exists and is defined in terms of f, it follows that f is not only continuously differentiable, but is also infinitely differentiable. □

LEMMA 3. *If f is continuous and satisfies (1), then $f''(x) = \lambda f(x)$ for some constant λ.*

PROOF. Proof: Differentiating (1) with respect to x twice, we get

$$f''(x+y) + f''(x-y) = kf''(x)f(y),$$

and differentiating (1) with respect to y twice, we get

$$f''(x+y) + f''(x-y) = kf(x)f''(y).$$

It follows that $f''(x)f(y) = f(x)f''(y)$. Let z be such that $f(z) \neq 0$, and substitute z for y in the above equation. This yields

$f''(x) = \frac{f''(z)}{f(z)} f(x)$, which shows that $f''(x) = \lambda f(x)$ for some constant λ. $\square$

Theorem 1. *The only continuous solutions to (1) are* $f(x) = 0$, $f(x) = \frac{2}{k}$, $f(x) = \frac{2}{k}\cos(\omega x)$, *and* $f(x) = \frac{2}{k}\cosh(\omega x)$.

Proof. The solutions to the differential equation in Lemma 3 are well known to be:

$$f(x) = mx + b, \ \ (m, b) \in \mathbf{R}^2, \text{ if } \lambda = 0,$$

$$f(x) = A\cos(\omega x) + B\sin(\omega x), \ \ (A, B) \in \mathbf{R}^2,$$

$$\omega^2 = -\lambda, \ \omega > 0 \text{ if } \lambda < 0, \text{ and}$$

$$f(x) = A\cosh(\omega x) + B\sinh(\omega x), \ \ (A, B) \in \mathbf{R}^2,$$

$$\omega^2 = \lambda, \omega > 0 \text{ if } \lambda > 0.$$

Since f is even and satisfies the the condition $f(0) = \frac{2}{k}$, the only possible solutions are $f(x) = 0$, $f(x) = \frac{2}{k}$, $f(x) = \frac{2}{k}\cos(\omega x)$, and $f(x) = \frac{2}{k}\cosh(\omega x)$, which finishes the proof. $\square$

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