Solutions and Discussions

Problem 3 — Volume 25, No. 2, Fall, 2001

To multiply binomials $(ax + b)(cx + d) = (ac)x^2 + (ad + bc)x + bd$, it initially appears as though 4 scalar multiplications will be needed. However, show how to multiply these binomials using only 3 scalar multiplications, plus any number of scalar additions and subtractions.

Solution

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Editor's Note: This solution was submitted by Nick Newman, now a graduate student at Auburn University, while he was a sophomore at Troy State University.

This operation can be performed using three multiplications in the following way:

$$x^2(ac) + x[(a+b)(c+d) - ac - bd] + bd$$

$$(ax+b)(cx+d) = x^2(ac) + x[(a+b)(c+d) - ac - bd] + bd$$

Editor's Note: The following submission addresses a claim made in the *Activities* section of the Spring 2004 (Vol. 28 No. 1) issue of the *Journal.* The excerpt which contains the claim appears below:

> When farmers Clarence and Myrtle died, they left two daughters their land with instructions to divide it equally. One daughter, Ella, was considerably brighter (and more conniving) than her sister, Jo. The land, unfortunately, was shaped as an irregular quadrilateral, and it wasn't immediately obvious how to divide it equally. Ella first tried

to get Jo to agree to split it down the diagonal AC shown, with Ella getting region ACD and Jo getting region ABC. Even Jo could see that was a bad deal, so she called her lawyer.

Ella then offered to spit the land with both diagonals. Ella would take two regions: AED and BEC, leaving Jo with ABE and CED. This sounded good to Jo, but her lawyer checked it out and reported that the sums of the areas of the respective regions were still not equal. "Ah," said Ella, "but the products of the areas of our regions *are* equal!" This stumped Jo and her lawyer and she agreed to the deal out of sheer awe for Ella's discovery.

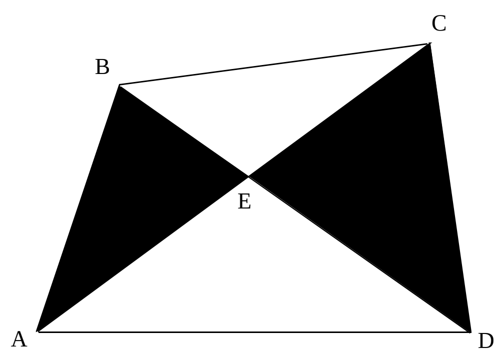

Figure 1

Solution

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Intuitively, the claim appears to be naive at best. However, a simple geometric proof, requiring nothing more than high school geometry, shows the claim to be true. To prove the claim, consider Figure 2 below:

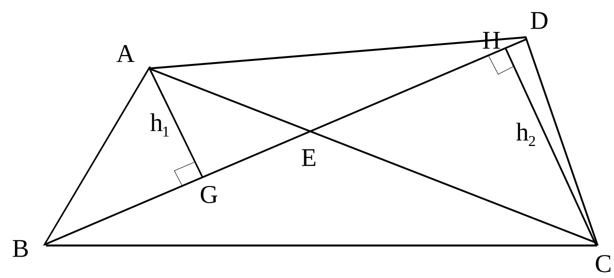

Figure 2

Choose point G on $\overline{AE}$ such that $\overline{AG}$ is perpendicular to $\overline{AE}$. Similarly, choose point H on $\overline{DE}$ such that $\overline{CH}$ is perpendicular to $\overline{DE}$.

Designate $\left|\overline{BE}\right| = b,\ \left|\overline{DE}\right| = d,\ \left|\overline{AG}\right| = h_1,\ \left|\overline{CH}\right| = h_2$.

Observe:

$$\begin{aligned}
\text{Area } \triangle ABE &= \tfrac{1}{2}\left|\overline{BE}\right|\left|\overline{AG}\right| = \tfrac{1}{2}bh_1\\
\text{Area } \triangle BCE &= \tfrac{1}{2}\left|\overline{BE}\right|\left|\overline{CH}\right| = \tfrac{1}{2}bh_2\\
\text{Area } \triangle CDE &= \tfrac{1}{2}\left|\overline{DE}\right|\left|\overline{CH}\right| = \tfrac{1}{2}dh_2\\
\text{Area } \triangle ADE &= \tfrac{1}{2}\left|\overline{DE}\right|\left|\overline{AG}\right| = \tfrac{1}{2}dh_1.
\end{aligned}$$

From these equations, it follows that

$$\begin{aligned}
(\text{Area } \triangle ABE)\,(\text{Area } \triangle CDE) &= \tfrac{1}{4}bdh_1h_2\\
(\text{Area } \triangle BCE)\,(\text{Area } \triangle ADE) &= \tfrac{1}{4}bdh_1h_2.
\end{aligned}$$

Thus we have:
$(\text{Area } \triangle ABE)\,(\text{Area } \triangle CDE) = (\text{Area } \triangle BCE)\,(\text{Area } \triangle ADE)\,.$

Our problem also lends itself to solution using vector analysis.

Recall that the area of a parallelogram with adjacent sides $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ is given by $||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}||\sin(\theta)$ (where θ is the angle with sides $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$). It follows that the area of a triangle with sides $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ is given by $\frac{1}{2}\,||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}||\sin(\theta)$.(See Figure 3).

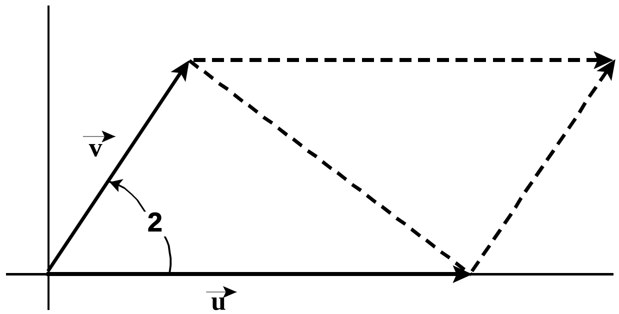

Figure 3

Also recall from analytic geometry that given vectors $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ with angle θ in between, that $\cos(\theta) = \frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}||} \Rightarrow ||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}|| = \frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{\cos(\theta)}$.

Hence, the area of a triangle with sides $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ is given by $\frac{1}{2}\,||\tilde{\mathbf{u}}||\,||\tilde{\mathbf{v}}|| \sin(\theta) = \frac{1}{2}\frac{\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}}}{\cos(\theta)} \sin(\theta) = \frac{1}{2}\tilde{\mathbf{u}} \circ \tilde{\mathbf{v}} \tan(\theta)$.

To solve our problem using vectors, consider Figure 4 below.

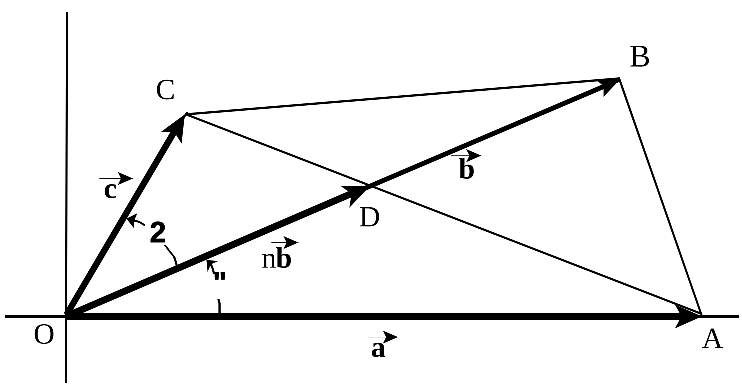

Figure 4

Designate $\overline{OA} = \tilde{\mathbf{a}}$, $\overline{OB} = \tilde{\mathbf{b}}$, and $\overline{OC} = \tilde{\mathbf{c}}$. The subtle part of our construction is that the line segments $\overline{OD}$ and $\overline{OB}$ are colinear and can, therefore, be written in terms of the same vector. Thus, we designate $\overline{OD} = n\tilde{\mathbf{b}}$ for some scalar n.

Observe:

$$\begin{aligned}
\text{Area } \triangle COD &= \tfrac{1}{2}\tilde{\mathbf{c}}\circ\left(n\tilde{\mathbf{b}}\right)\tan(\theta)\\
&= \tfrac{1}{2}n\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\tan(\theta)\\
\text{Area } \triangle COB &= \tfrac{1}{2}\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\tan(\theta)\\
\Rightarrow\ \text{Area } \triangle CDB &= (\text{Area } \triangle COB) - (\text{Area } \triangle COD)\\
&= \tfrac{1}{2}\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\tan(\theta) - \tfrac{1}{2}n\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\tan(\theta)\\
&= \tfrac{1}{4}n\,(n-1)\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\tan(\theta)
\end{aligned}$$

Also:

$$\begin{aligned}
\text{Area } \triangle ODA &= \tfrac{1}{2}\left(n\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha)\\
&= \tfrac{1}{2}n\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha)\\
\text{Area } \triangle OBA &= \tfrac{1}{2}\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha)\\
\Rightarrow\qquad \text{Area } \triangle BDA &= (\text{Area } \triangle OBA) - (\text{Area } \triangle ODA)\\
&= \tfrac{1}{2}\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha) - \tfrac{1}{2}n\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha)\\
&= \tfrac{1}{2}\,(1-n)\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha)
\end{aligned}$$

Finally: $(\text{Area } \triangle COA)\,(\text{Area } \triangle BDA)$

$$\begin{aligned}
&= \left[\tfrac{1}{2}n\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\tan(\theta)\right]\left[\tfrac{1}{2}\,(1-n)\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha)\right]\\
&= \tfrac{1}{4}n\,(1-n)\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\theta)\tan(\alpha)
\end{aligned}$$

And:

$$\begin{aligned}
&= (\text{Area } \triangle CBD)\,(\text{Area } \triangle ODA)\\
&= \left[\tfrac{1}{2}\,(1-n)\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\tan(\theta)\right]\left[\tfrac{1}{2}n\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\alpha)\right]\\
&= \tfrac{1}{4}n\,(1-n)\left(\tilde{\mathbf{c}}\circ\tilde{\mathbf{b}}\right)\left(\tilde{\mathbf{b}}\circ\tilde{\mathbf{a}}\right)\tan(\theta)\tan(\alpha)
\end{aligned}$$

i.e., $(\text{Area } \triangle COA)\,(\text{Area } \triangle BDA) = (\text{Area } \triangle CBD)\,(\text{Area } \triangle ODA)$