Problems

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(1) Find a simplified closed-form expression that is equal to this summation:

$$\sum_{j=0}^{n} j \binom{n}{j}.$$

(2) Compute the sum of the reciprocals of all the real and complex roots of this equation:

$$7x^6 + 24x^5 + 53x^4 + 97x^3 + 60x^2 + 18x + 5 = 0.$$

(3) Each weight comparison on a balance scale compares two objects and determines which is lighter and which is heavier. Given four objects with four distinct weights, show how to arrange these four objects in order from lightest to heaviest by performing only five weight comparisons.

(4) Let S_n denote the set of all one-to-one and onto functions $F : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Randomly select an arbitrary function F from the set S_n. Determine the probability P_n that we will have $F(k) \neq k$ for all k in the range $1 \leq k \leq n$. Also, determine $lim_{n\to\infty} P_n$.

(5) Suppose your friend chooses an integer value in the range from 1 to N. You may guess any value, and after each guess your friend must tell you whether your guess was "correct" or "too high" or "too low." Assuming you use the best possible guessing strategy, how many guesses are necessary to guarantee that you can guess the correct number if $N = 1000$? How many guesses are necessary if $N = 1,000,000$?

(6) Let the altitudes of a triangle have lengths a, b, and c. Inscribe a circle inside this triangle. Determine the radius of this circle as a function of a, b, and c.

(7) Suppose that there are N teams that compete in a round-robin basketball tournament; that is, each team plays one game against each of the other $N - 1$ teams. Prove that it will always be possible to label these N teams as $T_1, T_2, \ldots, T_N$ in such a way that team T_k defeats team T_{k+1} for $1 \leq k \leq N - 1$.