Lewis-Parker Lecture

On Being Close

By John C. Mayer

Abstract. Some examples of how the concept of being close has been "mathematicized" are discussed.

1. Introduction

How important is it to "mathematicize," or quantify, the idea that two things are close to each other? This paper does not purport to be an exhaustive treatment, but rather a brief introduction to encourage the reader to pursue those aspects of the idea he or she finds interesting. Presented below are three examples from current events which are stimulating, but which we will not discuss. The reader is invited to research what is speculative, conjectured, and provable about them.

- *Epidemics and Bioterrorism.* It is difficult and cost prohibitive to determine the precise pathways by which an infection has spread. (Some victims are too dead to interview; others haven't a clue or can't be found.) However, it may be possible to trace the pathways indirectly by determining the nucleotide sequence "distance" from the virus that infected one person to the virus that infected another. Could such an analysis help to determine how

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best to mount a vaccination effort in response to a bioterrorist attack? [**JK**]

- *Investment Portfolios.* It is desirable to know, particularly in a dubious investment climate, how to modify ones equity portfolio so as to move it closer to the best performing investments while maintaining diversity. Is there a mathematically defensible notion of "closer to the best performing" in this context?
- *Cloning and Chimeras.* Far more disturbing ethically than the cloning of human beings, is the possibility that human/animal chimeras may be created. Is a viable offspring possible when crossing, for example, humans and chimpanzees? The *Biology Workbench* is a computer research tool which purports to measure the "genetic distance" between organisms. According to the distance therein computed, humans and chimps are considerably closer than, for example, burros and horses, where viable (though sterile) crosses are well known (not to mention commercially viable). [**BW**]

What we will briefly survey are three examples of how mathematicians (broadly including logicians) have mathematicized the notion of being close. These examples are extracted from the fields of topology, logic, and graph theory, respectively.

2. Metrics: Distance from Euclid to Descartes

For Euclid and his predecessors among Hellenic and Hellenistic geometers, distance was *relational*. For example, in Figure 1, which shows similar triangles $ABC \sim ADE$, the classical statement $AB : AD :: AC : AE$ expresses the fact that the ratio of the length of AB to AD is the same as the ratio of the length of AC to AE. Taking AB as a unit, the other segments in the figure can all be related back to AB through this and other such proportions.

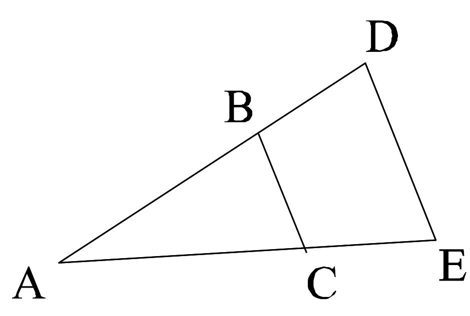


FIGURE 1. Similar triangles

But what *is* the length of AB? To answer this question, we jump ahead to Descartes. Assume we know what distance on a line is, given a unit length on that line. Assume that there are a pair of orthogonal coordinate axes (perpendicular lines) in the plane and a unit length has been chosen, the same on each axis. The point where the axes cross is the *origin* $0 = (0, 0)$. Denote the distance between points x_1 and y_1 on the first (horizontal) axis by $|x_1 - y_1|$, and similarly for the second (vertical) axis. Each point x in the plane has a pair of *Cartesian coordinates* $x = (x_1, x_2)$.

The *Euclidean distance* $D(x, y)$ between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in the plane is defined by $D(x, y)^2 = |x_1 - y_1|^2 + |x_2 - y_2|^2$. Note that this definition is consistent with the Pythagorean Theorem (see Figure 2).

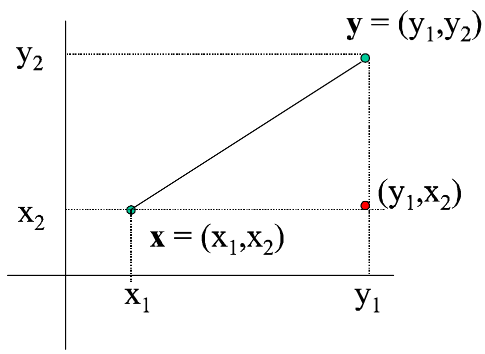
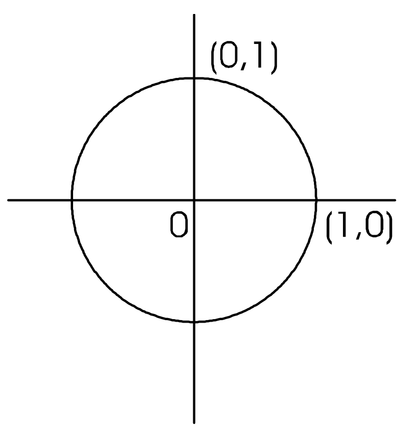


FIGURE 2. Euclidean distance

For analytical purposes, it is useful to define the *Euclidean ball* about a point c in the plane. The ball about *center* c of *radius* r is the set of points $B_D(c, r) = \{x \mid D(c, x) < r\}$. In Figure 3, we have illustrated the ball $B_D(0, 1)$.



Euclidean ball $B_D\,(0, 1)$

The ball does not include its boundary circle, so it is called an *open* ball. Using the language of balls, we can say that x and y are "close" if y is inside a "small" ball about x. How close depends upon how small, and small can be specified in terms of how the radius compares to the unit.

However, Euclidean distance is not always appropriate. For example, if I am going to walk from the Mathematics Department through the streets of Birmingham to the Fishmarket Restaurant for lunch, then my route might look like Figure 4.

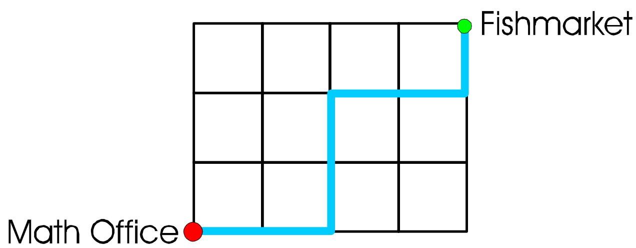


Figure 4. Lunch route

The Euclidean distance from department to restaurant is not the distance my feet experience! More appropriate, perhaps, is the *taxicab distance* from $x = (x_1, x_2)$ to $y = (y_1, y_2)$ defined by $T(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Another alternative distance is the *square distance*. The square distance from $x = (x_1, x_2)$ to $y = (y_1, y_2)$ is defined by $S(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Figure 5 illustrates the unit balls $B_T(0, 1)$ and $B_S(0, 1)$ for the taxicab and square definitions of distance, respectively, and Figure 6 shows how balls of the three distances about the same point as center compare. Inside any ball defined in terms of one of the distances, you can find a (possibly smaller) ball defined in terms of each of the other distances.

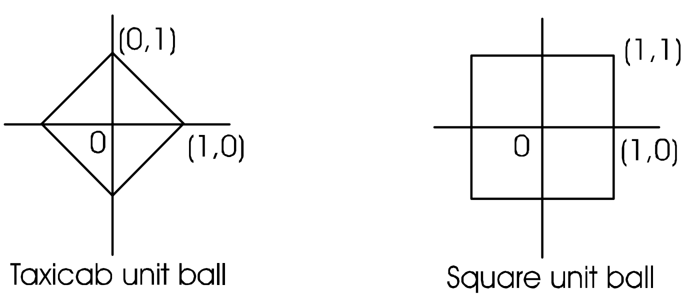


Figure 5. Unit balls $B\,(0, 1)$

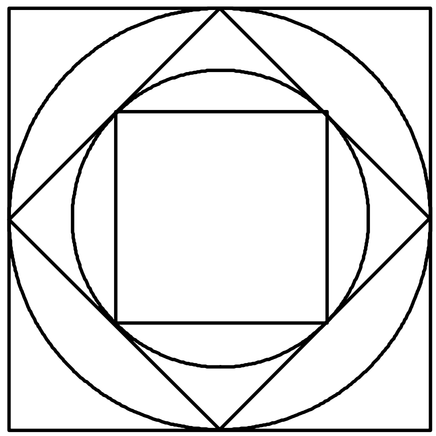


FIGURE 6. Nested balls

To continue our comparison of these three definitions of distance in the plane, we define the distance from a point to a set of points. Let A be a nonempty set of points in the plane and let x be a point. The Euclidean distance from x to A is defined by

$$D(x, A) = \inf\{D(x, a) \mid a \in A\}.$$

The taxicab and square distances to sets are defined similarly. The reader can easily prove the following theorem (see Figure 7).

Theorem[Limit Theorem] For each $r > 0$ there is a point of A in the ball $B_D(x, r)$, if, and only if, $D(x, A) = 0$.

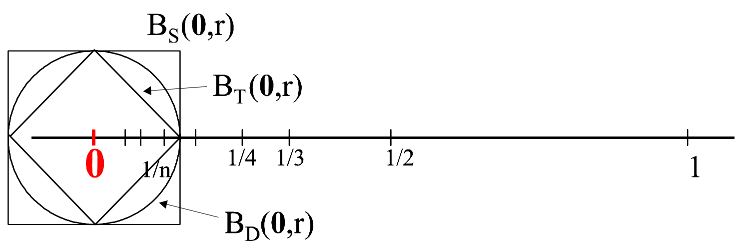


FIGURE 7. Limit point 0 of sequence $A = \{\frac{1}{n}\}_{n=1}^{\infty}$ is at distance 0 from A.

Two remarkable but distinct facts are

- The Limit Theorem remains true if we use, instead, the taxicab or the square distance definitions.
- For any x and A, all three definitions of distance agree on whether or not the distance from x to A is 0.

There is an underlying notion of being close that is common to the three definitions of distance given above. This underlying notion is the subject matter of metric topology, concerning which, the interested reader is referred to [**JM**]. It turns out that all three definitions of distance generate the same topology on the plane.

The key properties of any "distance" function D are the following:

- *Positive Definite*: $D(x, y) \geq 0$, and $D(x, y) = 0$ iff $x = y$.
- *Symmetric*: $D(x, y) = D(y, x)$.
- *Triangle Inequality*: $D(x, z) \leq D(x, y) + D(y, z)$.

We call any distance function with these three properties a *metric* on the space in consideration. Many useful theorems, such as the Limit Theorem above (which makes analysis possible), hold for metric spaces in general. The discipline of topology studies notions of closeness that are still more general than metrics. This has become a very prolific and useful branch of mathematics with many applications.

3. Counterfactual Conditionals: Spheres of Possible Worlds

Philosophical logicians have long debated the truth conditions for so-called *counterfactual conditionals*. Below are some examples of counterfactual conditionals. Suppose for the sake of argument that there is a perfectly good match sitting on the table (of our imagination).

(1) If that match were struck, it would light.
(2) If that match were soaked in water and then struck, it would not light.
(3) If that match were soaked in water and then struck, it would light.
(4) If kangaroos had no tails, they would topple over.

In all four conditionals, the antecedent is presumably false. If we were to use the truth conditions for the *material conditional* "if p, then q," which we learned in our first course in logic, then all four conditionals would be (vacuously) true. Intuitively, however, conditionals (1), (2), and (4) above are true and (3) is false. We perceive a connection of some sort between antecedent and consequent which determines the truth or falsity of the conditional.

In his book *Counterfactuals*, the Princeton philosopher David K. Lewis, proposed analyzing the truth conditions for such conditionals in terms of proximity of possible worlds [**DL**]. From a purely formal point if view, a possible world is just a collection of all the propositions taken to be true. But we will adhere to the slightly more metaphysical position of speaking of possible worlds as entities *at which* propositions are true or false. The real world is one of these entities, as are the (unrealized) worlds in which the match is struck, rather than just sitting there on the table of our imagination.

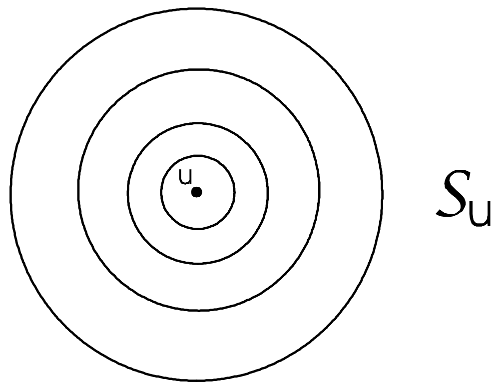


FIGURE 8. System $\mathcal{S}_u$ of spheres of possible worlds about world u.

Following Lewis, each possible world u is the *center* of a *system* $\mathcal{S}_u$ *of spheres* of possible worlds satisfying the following conditions (see Figure 8):

- *Centered.* $\{u\}$ is a sphere in $\mathcal{S}_u$.
- *Nested.* If S and T are in $\mathcal{S}_u$, then one is a subset of the other.
- $\mathcal{S}_u$ is closed under unions and (nonempty) intersections.

According to Lewis, "The system of spheres used in interpreting counterfactuals is intended to carry information about the comparative overall similarity of worlds." Assume that *comparative overall similarity* is a notion that, though undefined, we understand how to apply. There may be ties in comparative overall similarity. Consider Lewis' example of the following counterfactuals.

(1) If Bizet and Verdi were compatriots, they would both be French.
(2) If Bizet and Verdi were compatriots, they would both be Italian.
(3) If Bizet and Verdi were compatriots, they would both be German.
(4) If Bizet and Verdi were compatriots, they would both be French or Italian.

It seems to me that worlds where Bizet and Verdi were compatriots and both French would be in the same sphere S_1 about the real world as worlds where Bizet and Verdi were compatriots and both Italian. But I think worlds where they are both German are further out, in a sphere S_2 properly containing S_1. It is even arguable that there is a sphere of worlds S_0 properly contained in S_1 where they are compatriots, and both French or Italian. Consequently, we might well evaluate (4) as true, and (1), (2), and (3) all as false, in the real world. (Here, the interested reader might look up the "Law of the Excluded Middle," and ponder whether or not it holds for counterfactuals.)

With some additional terminology, we can define the Lewis truth conditions for counterfactuals. Let p be a proposition. Let $\overline{p}$ denote its negation. A *p-world* is a possible world at which p is true. A *p-permitting sphere* is a sphere which contains at least one p-world. The truth conditions for a counterfactual can now be stated as:

Definition The counterfactual "if p were true, then q would be true," symbolized $p \Rightarrow q$, is *true* at world u iff either no p-world belongs to any sphere in $\mathcal{S}_u$, or some sphere S in $\mathcal{S}_u$ contains a p-world, and q holds at each p-world in S.

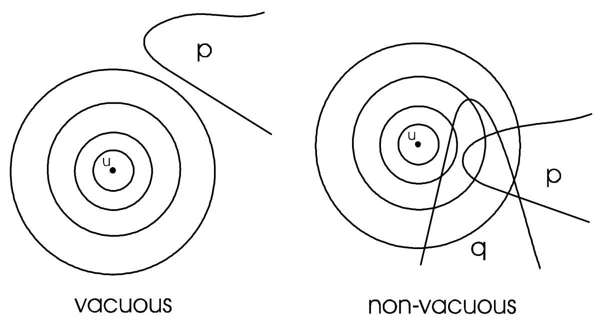


Figure 9. Vacuous and non-vacuous truth of the counterfactual $p \Rightarrow q$ at world u.

See Figure 9 illustrating the vacuous and non-vacuous truth of $p \Rightarrow q$ at world u. For vacuous truth, $p \Rightarrow q$ and its *opposite* $p \Rightarrow \overline{q}$ both hold. For non-vacuous truth, $p \Rightarrow q$ and $\overline{p \Rightarrow \overline{q}}$ both hold. Counting u as the zeroth sphere in Figure 9, the third sphere out from u is the key one for seeing that $p \Rightarrow q$ is true in the non-vacuous case. There are no p-worlds in the second sphere, though some are q-worlds. There are p-worlds which are not q-worlds in the fourth sphere. These truth conditions clearly differ from those for the corresponding material conditional.

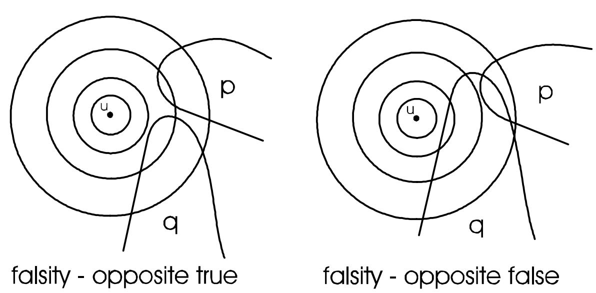


Figure 10. Two ways that counterfactual $p \Rightarrow q$ can be false at world u.

Figure 10 illustrates cases where $p \Rightarrow q$ is false, contrasting the case where the opposite, $p \Rightarrow \overline{q}$, is true with the case where the opposite is also false.

Is there a distance between possible worlds? To answer this question, the reader has to consider whether his or her notion of comparative overall similarity of possible worlds is positive definite, symmetric, and satisfies the triangle inequality. Consideration of the Bizet/Verdi counterfactuals and Figure 11 may be helpful in this task, which we leave to the reader.

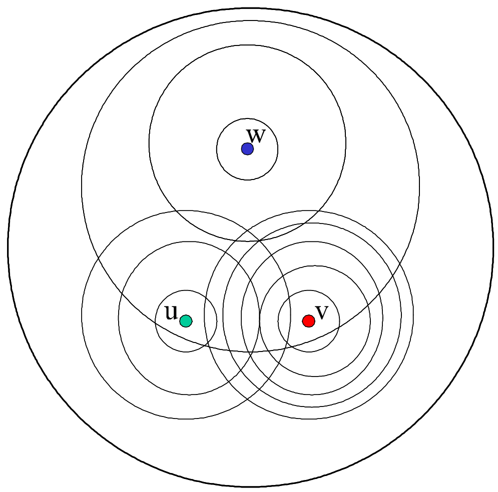


Figure 11. Is comparative overall similarity metrizable?

The purpose of Lewis' analysis of counterfactuals in terms of spheres of possible worlds is not to quantify truth conditions of counterfactuals precisely, but rather to elucidate our qualitative understanding of what counterfactuals are. In philosophical logic, his was a seminal book of the 20th century.

4. Your Erdös Number: Small Worlds Graphs

How close are you to the center of the mathematical universe? Paul Erdös, well-known mathematician, wrote over 1,400 papers, and had over 500 co-authors. His centrality is indisputable. Yours can be measured perhaps by determining your *Erdös number.* If you are Erdös, your Erdös number is 0. If you wrote a joint paper with Erdös, your Erdös number is 1. If you did not write a paper with Erdös, but wrote one with an Erdös co-author, your Erdös number is 2. ... And so on.

My Erdös number is 3, an honor I share with 26,422 other mathematicians (more or less). The chain from me to Erdös is

Erdös $\longleftrightarrow$ D. Mauldin $\longleftrightarrow$ B.L. Brechner $\longleftrightarrow$ Mayer

About $208,000$ mathematicians have a finite Erdös number, and about $130,000$ have an infinite Erdös number. The distribution of Erdös numbers, and much else of interest, can be found at the Erdös Number Project [**JG**]. Most of the statistics in this section are from this website.

In order to study the closeness of mathematicians more generally, let us construct a graph whose vertices are published mathematicians. Insert an edge between two mathematicians if they have been mathematical co-authors. There is one large (connected) component of about $208,000$ vertices, including Erdös. However, $84,000$ mathematical authors have written no joint papers. This construct is called the *collaboration graph.*

The *size* of a graph is the number of vertices. The *degree* of a vertex in a graph is the number of edges that contain that vertex. Since each edge contains two vertices, the only graph each of whose vertices is degree two is a circular chain. If all the vertices of a graph have the same degree, we call the graph *regular.* If a graph is regular and contains a circular chain through all its vertices, we call it *circulant.* The *average degree* of a graph is the average of the degrees at each vertex, taken over all vertices. A graph whose average degree is two need not be a circular chain. If a graph is created/selected by a random process (the details need not concern us), we will call it a *random graph.* How do regular graphs and random graphs of the same average degree compare?

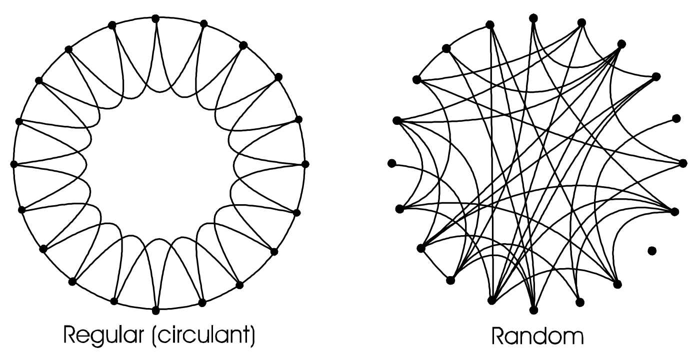


FIGURE 12. Regular (circulant) and random graphs of the same size (20) and average degree (4).

In Figure 12, we have illustrated a circulant (regular) graph with 20 vertices, each of degree 4, and a random graph (I rolled the dice myself) with 20 vertices and the same number (40) of edges (so average degree 4). Is the mathematics collaboration graph more like a regular graph or a random graph? The "closeness" of mathematicians is neither regular nor random, so what is the appropriate model?

Two measures of closeness associated with graphs are the *(average) path length* and the *(average) clustering coefficient*. Let v and w be vertices in a graph G. The *path length* $L(v, w)$ from v to w is the least number of edges it takes to get from v to w. The path length is ∞ if there is no path from v to w. In a connected graph, the path length between vertices is always finite. (At this point, the reader might check if, on the set of vertices of a connected graph G, $L(v, w)$ is a metric.) The *average path length* $\overline{L(v)}$ *at* v is the average of the path lengths $L(v, w)$ between v and all vertices w different from v in the graph. The *average path length* $\overline{L}$ of G is the average of $L(v, w)$ over all pairs of different vertices v and w in G. In Figure 13, the average path length from v is approximately 1.69.

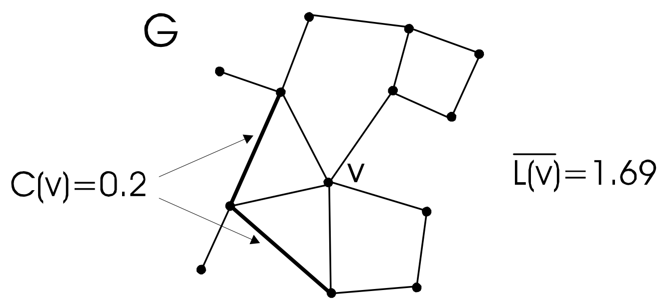


FIGURE 13. Average path length $\overline{L(v)}$ and clustering coefficient $C(v)$ at vertex v in graph G.

For circulant graphs of fixed degree, the average path length scales with the size. But for connected random graphs of fixed average degree, it scales with the log of the size [**DW**]. So, other things being equal, in regular graphs vertices are more spread out overall than in random graphs. The average path length for the regular graph in Figure 12 is approximately 2.89, while the average path length for the random graph, ignoring the isolated vertex, is very nearly 2. The difference would be larger if the number of vertices increased.

The *clustering coefficient* $C(v)$ of vertex v in graph G is the ratio of edges among v's immediate neighbors to the total possible number of such edges. For example, if v had five immediate neighboring vertices (connected to v directly by an edge), then the maximum number of edges among those five vertices that could exist is $C_2^5 = \frac{5*4}{2} = 10$. The clustering coefficient for vertex v in Figure 13 is thus $C(v) = \frac{2}{10} = 0.2$. The *average clustering coefficient* $\overline{C}$ is the average of $C(v)$ over all vertices v in G. For circulant graphs, the average clustering coefficient is independent of size, and approaches a constant as the degree increases [**DW**].

The average clustering coefficient for the circulant graph in Figure 12 is 0.5 while the average clustering coefficient for the random graph in Figure 12 is less than 0.05.

What kind of graph is the collaboration graph of mathematicians, random or regular? Experimentally, the average path length of the connected component of the collaboration graph is comparable to random graphs of the same size and average degree, while the clustering coefficient is comparable to regular graphs of the same average degree. So it is neither nearly random nor nearly regular.

Duncan J. Watts in his book *Small Worlds* [**DW**] identifies a class of graphs he calls *small worlds graphs* which have average path length comparable to random graphs and average clustering coefficient comparable to regular graphs.

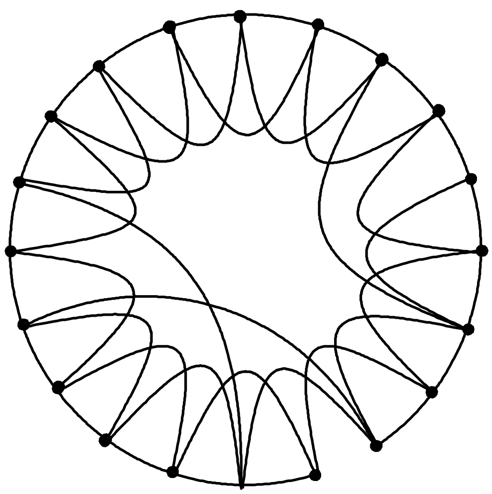


FIGURE 14. Small worlds graph of size 20 and average degree 4.

A model of a small worlds graph can be obtained by taking a circulant graph like that in Figure 13 and rewiring a few edges at random. We obtain a graph with the same number of vertices and edges as before rewiring (see Figure 14). As a result of the rewired "shortcuts," the average path length becomes comparable to that of a random graph, $\overline{L} << 2.89$, while the clustering remains like that of the corresponding regular graph, $\overline{C} \approx 0.5$.

Small worlds graphs are currently a topic of interest in modelling disease transmission, computer networks, social networks, and other systems where large numbers of discrete agents are connected to, and influence, each other along relatively short paths with a high degree of clustering. Ultimately, whether this notion is scientifically useful, I suspect, will be determined by whether any unexpected predictions can be made about a system based upon its being modelled by a small worlds graph. The jury us still out.

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