Solutions and Discussions

Problem 1 — Volume 26, No. 1, Spring, 2002

Compute $10^{1024} \operatorname{mod} 23$ *without using a calculator or a computer. [Hint: use repeated squaring.]*

Solution

Jimmy Ma, Sophomore, Loveless Academic Magnet High School, Montgomery, AL.

Our solution hinges on the following observation. Let $N = (23x+y)^n$. Then the remainder of N when divided by 23 is the same as the remainder of y^n when divided by 23.

To see this, we expand $N = (23x+y)^n$ using the binomial formula:

$$N = (23x+y)^n = (23x)^n + n\,(23x)^{n-1}\,y^1 + \tbinom{n}{2}\,(23x)^{n-2}\,y^2 + \ldots$$
$$+n\,(23x)\,y^{n-1} + y^n$$

Next, we express y^n as $y^n = 23m + r$, as delineated by the division algorithm. Thus, we can write:

$$N = (23x+y)^n = (23x)^n + n\,(23x)^{n-1}\,y^1 + \tbinom{n}{2}\,(23x)^{n-2}\,y^2 + \ldots$$
$$+n\,(23x)\,y^{n-1} + 23m + r = 23k + r.$$

Using our observation, and applying it to 10^{1024}, we have:

$$10^{1024} = \left(10^2\right)^{512} = 100^{512} = (23\cdot 4+8)^{512} = 8^{512}\,(\operatorname{mod} 23)\,.$$

But $8^{512} = \left(8^2\right)^{256} = 64^{256} = (23\cdot 2+18)^{256} = 18^{256}\,(\operatorname{mod} 23)\,.$

And $18^{256} = \left(18^2\right)^{128} = 324^{128} = (23\cdot 14+2)^{128} = 2^{128}\,(\operatorname{mod} 23)\,.$

Note that $2^{128} = \left(2^8\right)^{16} = 256^{16} = (23\cdot 11+3)^{16} = 3^{16}\,(\operatorname{mod} 23)\,.$

Also, $3^{16} = \left(3^4\right)^4 = 81^4 = (23 \cdot 3 + 12)^4 = 12^4 \,(\mathrm{mod}\, 23)\,.$

But $12^4 = \left(12^2\right)^2 = 144^2 = (23 \cdot 6 + 6)^2 = 6^2 \,(\mathrm{mod}\, 23)\,.$

Finally, $6^2 = 36 = (23 \cdot 1 + 13) = 13 \,(\mathrm{mod}\, 23)\,.$

Thus, we have:$10^{1024} = 8^{512} \,(\mathrm{mod}\, 23) = 18^{256} \,(\mathrm{mod}\, 23) = 2^{128} \,(\mathrm{mod}\, 23) = 3^{16} \,(\mathrm{mod}\, 23) = 12^4 \,(\mathrm{mod}\, 23) = 6^2 \,(\mathrm{mod}\, 23) = 13 \,(\mathrm{mod}\, 23)$

So $10^{1024} \bmod 23 = 13$

Also solved by Luay Abdel-Jaber, Auburn University at Montgomery, Montgomery, AL.

Problem 2 — Volume 26, No. 1, Spring, 2002.

Suppose you have one large chocolate bar that consists of 24 *small squares arranged in the shape of a* 4*-by-*6 *matrix. Determine the minimum possible number of times you will need to break the bar so that each of the* 24 *small squares is separated from all the others. You may only break one bar at a time, and you may only break each bar along a straight line. Also try to generalize your answer by considering a chocolate bar that consists of xy small squares arranged in the shape of an x-by-y matrix.*

Solution

Nick Newman, Junior, Troy State University, Troy, AL.

Given an $x \times y$ matrix A, let $n = xy$ (i.e., A has n elements)..The rules of the problem allow for breaks along a joint between rows or a joint between columns.

Claim 1. *If matrix A consists of n elements, then there exist exactly $n-1$ breaks. In other words, if $B(n)$ denotes the number of breaks required to completely break up the candy bar matrix, then $B(n) = n-1$.*

Proof. We prove this by induction on n. Our induction hypothesis,$P(n)$, is the proposition that $B(n) = n-1$. For cases $n = 1, 2$, our claim is obviously true. If $n = 1$, then there are no breaks possible, and $B(1) = 1 - 1 = 0$. If $n = 2$, then there is exactly one break possible, and $B(2) = 2 - 1 = 1$.

Next, for the induction step, assume that $P(k)$ is true for $k \in S = \{1, 2, \ldots n-1\}$. We show that $P(k)$ also holds for $k = n$. Since our proposition $P(n)$ is true for $n = 2$, we assume, without loss of generality, that there are at least two rows (The proof for the case in which there are at least two columns is identical). By arbitrarily making our first break after the d^{th} row, we create two new, smaller matrices of sizes $(x-d)\times y$ and $d\times y$. Since $(x-d)\cdot y \in S = \{1, 2, \ldots n-1\}$, and $d\cdot y \in S = \{1, 2, \ldots n-1\}$, it follows that $P\left((x-d)\,y\right) = (x-d)\,y - 1$, and $P\left(dy\right) = dy - 1$. Adding these together, and taking into our consideration our initial break, we get:

$$P\left((x-d)\,y\right) + P\left(dy\right) + 1 = \left[(x-d)\,y - 1\right] + \left[dy - 1\right] + 1 = xy - 1$$

Thus, the proposition, $B\left(n\right) = n - 1$ holds for all $n \in \mathbf{N}$. $\square$

This shows that the number of breaks is independent of the way that the bar is broken. If $n = 4 \cdot 6 = 24$, then the minimum number of breaks is $24 - 1 = 23$. Furthermore, if $n = xy$, then the minimum number of breaks is $xy - 1$.

Problem 4 — Volume 26, No. 1, Spring, 2002.

Evaluate each of the following limits. [Note: you can grind through the messy details, or you can apply general concepts to solve these in your head.]

(1) (a) $\lim_{n\to\infty} \frac{n^2-9999n-99999999}{999999999+9999n+n^2}$

(b) $\lim_{n\to\infty} \frac{2^n}{\log n}$

(c) $\lim_{n\to\infty} \frac{\log^3 n}{2^{2^n}}$

Solution

Luay Abdel-Jaber, Auburn University at Montgomery, Montgomery, AL.

(1) (a) $\lim_{n\to\infty} \frac{n^2-9999n-99999999}{999999999+9999n+n^2}$

Since the terms of highest degree dominate the numerator and denominator, as $n \to \infty$, we can replace the terms of the numerator and denominator by the terms of highest degree and let $n \to \infty$. Hence,

$$\lim_{n\to\infty} \frac{n^2-9999n-99999999}{999999999+9999n+n^2} = \lim_{n\to\infty} \frac{n^2}{n^2} = 1$$

(b) $\lim_{n\to\infty} \frac{2^n}{\log n}$

Note that the numerator and the denominator are differentiable, and that the quotient satisfies the hypotheses of L'Hôpital's Rule. Thus, by applying that rule, we have:

$$\lim_{n\to\infty} \frac{2^n}{\log n} = \lim_{n\to\infty} \frac{2^n \ln 2}{\left(\frac{\ln(10)}{x}\right)} = \lim_{n\to\infty} x2^x \log(2) = \infty$$

(c) $\lim_{n\to\infty} \frac{\log^3 n}{2^{2^n}}$

To simplify our calculations, we shall prove that $\lim_{n\to\infty} \frac{\log n}{2^{\frac{2^n}{3}}} = 0$, from which it follows, by cubing the original function, that $\lim_{n\to\infty} \frac{\log^3 n}{2^{2^n}} = 0$.

To prove the claim that $\lim_{n\to\infty} \frac{\log n}{2^{\frac{2^n}{3}}} = 0$, we note that the hypotheses of L'Hôpital's Rule are satisfied. Hence,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log n}{2^{\frac{2^n}{3}}} &= \lim_{n\to\infty} \frac{\left(\frac{\ln(10)}{n}\right)}{\frac{1}{3}2^{\frac{1}{3}2^n+n}(\ln 2)^2} \\
&= \lim_{n\to\infty} \frac{3\ln(10)}{n2^{\frac{2^n}{3}+n}(\ln 2)^2} \\
&= 0
\end{aligned}$$

Thus, the claim is proved, and it follows that $\lim_{n\to\infty} \frac{\log^3 n}{2^{2^n}} = 0$.

Also solved by Sheena, Richards, Junior, Troy State University, Troy, AL.

Problem 5 — Volume 26, No. 1, Spring, 2002.

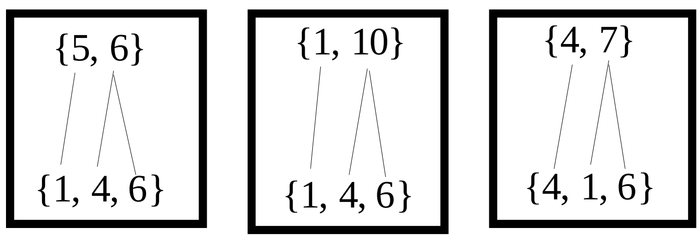
A simple game begins with 11 *stones arranged in a single pile. Two players take alternating turns. Each turn consists of selecting any pile that contains at least* 3 *stones, and then splitting this pile into two smaller piles. The only restriction is that, after each turn, all the currently remaining piles must contain different numbers of stones. The game ends when one of the players can make no legal move, and this player is declared to be the loser. Assuming that both players want to win the game, what should be the strategy of the first player on his/her first turn?*

Solution

Sheena Richards, Sophomore, Troy State University, Troy, AL

There are only two ways that eleven stones can be broken down into smaller piles, leaving no possible moves. One of these yields three final piles of size $1, 4, 6$, and the other yields four piles of sizes $\{1, 2, 3, 5\}$.

On his/her first move, Player 1 initially creates two piles. By taking his/her first move, Player 2 creates three piles. Player 1 does not want Player 2 to have the opportunity to create piles of sizes $\{1, 4, 6\}$. Piles of sizes $\{1, 4, 6\}$ can only be created from the two piles left by Player 1, in the following ways, shown below:



Thus, the strategy of Player 1, on his/her first turn, will be to avoid breaking the pile of eleven pebbles into two piles of sizes $\{5, 6\}$, $\{1, 10\}$, $\{4, 7\}$. Player 1 can win — guaranteed, by breaking the pile of eleven pebbles into two piles of sizes $\{5, 6\}$ or $\{1, 10\}$.