A Template for the Definition of Limit

By Don Alexander

The list of those that grappled with the issue of adequately defining "limit" reads like a Who's Who of the discipline. [2]
Prof. Steven A. Douglass

In most AP Calculus and college-level Calculus I courses, when teachers introduce limits, they typically investigate "nice" functions in the plane, frequently neglecting essential properties of their domains. "Pathological functions" or "sophisticated properties" of the number line are deferred for more advanced studies. Currently, computer algebra systems allow us to present a myriad of scenarios, affording our students the opportunity to explore the behavior of functional images in specific regions of the domain.

What Happens When the Formal Definition Arrives?

During the assessment of the limit definition, typically there is a somewhat bipolar distribution, with noticeable disparity in correctness, even in simply stating the definition. In more advanced courses, such a disparity of the definition, in the recitation as well as the depth of understanding, persists. The *template* that follows provides an objective format for presenting the definition, for considering each of the essential components, as well as assessment.

In the mid 1800s, the "father of modern analysis," Karl Weierstrass (as a high school teacher!) provided the "epsilon-delta" definition for limit, although it was not publicly revealed until 1859 while he was a professor at the University of Berlin. His syntax is essentially what appears in today's calculus texts. The ε-δ definition of Weierstrass is the basis for the *template* that follows.

In developing his theory, Weierstrass advocated a two-part program (named the "arithmetization of analysis" by Felix Klein) "…wherein the real number system itself should first be logically developed, and then the limit concept, continuity, differentiability, convergence and divergence, and integrability defined in terms of this number system." [4] The limit concept in introductory calculus does use certain nice subsets of real numbers — intervals. Accordingly, a portion of the *template* involves real intervals as subsets of the domain of a function. One must be sure that one **can** get "close enough!"

Sometimes, the concepts of "unboundedly large (small)," "unbounded," "approaches from above (below)," or "approaches from the left (right)" are involved. This author abhors the use of an equality sign with the symbol for infinity. However, such misuse of the equality sign is standard in calculus texts. Accordingly, with chagrin, such notation appears in the template. The use of " $\rightarrow$ " for "approaches" is also accepted for this template. It might be noted that, in several cases, the existence of p and q with $p < q$ is not necessary. However, in order to provide a general template, such existence is included.

The Template

Assume that both p and q are real numbers with $p < q$. Suppose that f is a function in the plane and $\_//C//\_$ is in its domain. The statement

$$\lim_{x \to \backslash\backslash A \backslash\backslash} f(x) = \backslash\backslash B \backslash\backslash$$

means that if $\varepsilon > 0$, then there is a $\delta > 0$ for which

$f(x)$ satisfies $\underline{\qquad\qquad //D// \qquad\qquad}$

whenever

x satisfies $\underline{\qquad\qquad //E// \qquad\qquad}$

Template Options

The options for region $\backslash\backslash A \backslash\backslash$ are a, a^+, a^-, ∞, $+\infty$, $-\infty$

for region $\backslash\backslash B \backslash\backslash$ are L, L^+, L^-, ∞, $-\infty$, or $+\infty$

for region $//C//$ are $(p, a) \cup (a, q)$, (p, a), (a, q), $(-\infty, p)$, $(q, +\infty)$, or $(-\infty, p) \cup (q, +\infty), (p, q)$

for region $//D//$ are $|f(x) - L| < \varepsilon$, $0 \leq L - f(x) < \varepsilon$, $0 \leq f(x) - L < \varepsilon$, $|f(x)| > \varepsilon$, $f(x) > \varepsilon$, or $f(x) < -\varepsilon$

for region $//E//$ are $0 < |x - a| < \delta$, $0 < x - a < \delta$, $0 < a - x < \delta$, $|x| > \delta$, $x > \delta$ or $x < -\delta$.

Students "supply" the appropriate options for regions // C //, // D //, and // E //, based on what is "teacher-supplied" in regions \\ A \\ and \\ B \\.

It should be noted that not only may ε and δ be used as tolerances of "small" or "close," they may also be used as standards for "large." Additionally, at least a cursory discussion of the logical conditional form, "*if ... then ...,*" is essential. There are 36 variations available in this template framework. The student-supplied responses can easily appear as "matching" from a list, or as "fill-in-the-blanks." This author's students seem to be more successful with the "matching" option. Even in such an objective format, absolute valued inequalities seem persistently difficult. With minor modifications, "higher" levels of cognitive development may be assessed using the template. Further, although the "one sided" limit properties are lost, replacing absolute values with Euclidean distance allows examining limits in n-space.

References

[1] R. Calinger, ed., *Classics of Mathematics,* Moore Publishing Company, Inc., 1982.

[2] S. Douglass, *Introduction to Mathematical Analysis,* Addison, Wesley, Longman, Inc., 1996.

[3] H. Eves, *Great Moments in Mathematics Before 1650,* The Mathematical Association of America, 1983.

[4] H. Eves, *Great Moments in Mathematics After 1650,* The Mathematical Association of America, 1983.

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