An Analytic Aspect of the Fibonacci Sequence

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The *Fibonacci numbers* F_n can be defined by the three-term linear recurrence equation

$$F_{n+2} = F_{n+1} + F_n \tag{1}$$

subject to the initial conditions $F_0 = 0$, and $F_1 = 1$.

It is evident that the terms of the Fibonacci sequence F_n are uniquely determined by (1) and the given initial conditions.

The most commonly known representation of the Fibonacci numbers in a closed form is attributed to the French mathematician Jacques-Philippe-Marie Binet (1843), and is called *Binet's formula.* It states that *the* n^{th} *Fibonacci number* can be written as

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \tag{2}$$

Establishing equation (2) is equivalent to solving equation (1) with the given initial conditions. For the convenience of the reader, let us make a quick review of the technique used to solve equation (1) with the given initial conditions. The reader may recognize that this technique is analogous to the one used for finding particular solutions of second-order linear ordinary differential equations with constant coefficients.

First, we associate a quadratic equation, called the *characteristic equation,* with (1), namely

$$r^2 = r + 1 \tag{3}$$

or, equivalently,

$$r^2 - r - 1 = 0. \tag{4}$$

Then, we find the solutions to (4),

$$a = \frac{1+\sqrt{5}}{2}, \ b = \frac{1-\sqrt{5}}{2}, \tag{5}$$

i.e., $\qquad a^2 - a - 1 = 0$ and $b^2 - b - 1 = 0$.

If we multiply the preceding equations by the powers a^n and b^n, respectively (where n is an arbitrary natural number), we see that each of these powers will also be a solution of the recurrence formula (1), with a^n and b^n playing the role of F_n. Moreover, by linearity, any linear combination

$$F_n(\alpha, \beta) = \alpha a^n + \beta b^n \tag{6}$$

of a^n and b^n will also be a solution of (1), and, in fact, (6) can be considered to be the "general solution" of (1). Now, it is a matter of algebra to determine α and β so that

$$\begin{aligned} F_0(\alpha, \beta) &= 0 \\ F_1(\alpha, \beta) &= 1 \end{aligned} \tag{7}$$

are simultaneously satisfied, and this is the case if $\alpha = \frac{1}{\sqrt{5}}$, and $\beta = -\frac{1}{\sqrt{5}}$, whence (2). (For all of these, see, e.g., [2].)

This deduction is very simple, entirely elementary, and algebraic. As a matter of fact, we have used nothing from the theory of differential equations except for the notions of the characteristic polynomial, the particular solution, and the general solution — notions lying well within the realm of linear algebra by the "discrete nature" of difference equations. So, the acclaimed intimacy between the theory of linear difference equations and that of linear ordinary differential equations, at least to the extent discussed here, appears to be purely *algebraic.*

Recently, we came across the paper, "Introduction to Analytic Fibonometry" by Robyn Minor Smith [1]. Though the paper was written with different objectives in mind, it raised the question of an *analytic* feature in the relationship between the two theories that seems to have been overlooked.

In her paper, in analogy to (1), Smith obtained the particular solution $y_0(x)$ of the differential equation

$$y'' - y' - y = 0 \tag{8}$$

subject to the initial-value conditions, $y_0(0) = 0$, $y_0'(0) = 1$, in the usual way. With the values of a and b in (5), this turns out to be

$$y_0(x) = \frac{e^{ax} - e^{bx}}{\sqrt{5}} = \sum_{n=0}^{\infty} \frac{a^n - b^n}{\sqrt{5}n!} x^n. \tag{9}$$

On the other hand, Smith also derived a particular solution in terms of a Maclaurin series with undetermined coefficients, i.e., starting with the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \tag{10}$$

with $c_0 = 0$, and $c_1 = 1$. (It is classical that such a series solution exists with an infinite radius of convergence.) In doing this, she was essentially led to the following recurrence formula for the undetermined coefficients c_n :

$$(n+2)!c_{n+2} = (n+1)!c_{n+1} + n!c_n \qquad (c_0 = 0,\ c_1 = 1)\,. \tag{11}$$

Hence, she made the observation that the sequence $\{n!c_n\}$ satisfies the same recurrence equation as the Fibonacci sequence, so $n!c_n = F_n,\ n \geq 0$, i.e.,

$$c_n = \frac{F_n}{n!} \qquad (n = 0, 1, 2, \ldots)\,. \tag{12}$$

Therefore, the Maclaurin Series

$$y(x) = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n \tag{13}$$

is an *analytic* solution of (8) satisfying the same initial-value conditions as $y_0(x)$. Consequently, by the *uniqueness* part of the fundamental theorem of existence and uniqueness of solutions of second-order ordinary differential equations, the analytic function $y(x)$ represented in (13) is identical to $y_0(x)$ given in (9). This also tells us that the analytic function $y(x)$ is an elementary function, and, indeed, we have

$$\frac{e^{ax} - e^{bx}}{\sqrt{5}} = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n. \tag{14}$$

In Theorem 1 ([1]), Smith appears to take identity (14) for granted. We think that the preceding argument could be a way to fill this gap. Anyway, it would be rather difficult to establish identity (14) directly, without making use of the theorem of existence and uniqueness of solutions of ordinary differential equations.

The appearance of the Fibonacci numbers in the Maclaurin series of the particular solution $y_0(x)$ of (8) is quite surprising and can be considered as a further addition to the parallel between the theory of linear difference and differential equations, but this time from a purely analytic viewpoint. As a bonus, we also get a nonelementary proof of Binet's formula. In fact, since the expansion of an analytic function into a power series is unique, the coefficients of like terms in (9) and (14) must be identical. Thus, a straightforward comparison of the coefficients will yield (2).

The same observations apply, e.g., to the *Lucas numbers* L_n defined by the linear recurrence

$$L_{n+2} = L_{n+1} + L_n \tag{15}$$

subject to $L_1 = 1$, and $L_2 = 3$. Arguing as above, we may deduce

$$e^{ax} + e^{bx} = \sum_{n=0}^{\infty} \frac{a^n + b^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{L_n}{n!} x^n, \tag{16}$$

and then obtain a closed form for the Lucas numbers:

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n \geq 0. \tag{17}$$

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References

[1] Robyn Minor Smith, *Introduction to Analytic Fibonometry,* Alabama Journal of Mathematics, 25:2 (2001), 27-36.

[2] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers,* 5th Edition, John Wiley & Sons, 1991.

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