Parabolic Tangents: An Algebraic Analysis

By David R. Duncan and Bonnie H. Litwiller

Abstract. Teachers are always seeking interesting problem solving situations that employ a variety of mathematics content areas. We shall describe one such situation in which analytic geometry and calculus are both central to the solution of the problem.

The Problem

Given a circle and a point in its exterior, it is always possible to construct two tangents from the point to the circle. Given a parabola and a point in its exterior, how many tangents to the parabola through the given point can be constructed?

Figure 1 depicts a parabola $y = kx^2$ with $k > 0$.

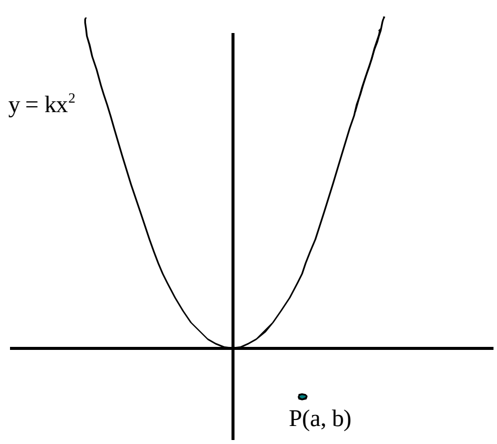

Figure 1

The exterior region is represented by $y < kx^2$; the interior region is represented by $y > kx^2$; and the set of points on the parabola is represented by $y = kx^2$.

Let $P(a,b)$ be a point not on the parabola and let $Q(c,d)$ be a point on the parabola, so that line $\overleftrightarrow{PQ}$ is tangent to the parabola, $y = kx^2$ at Q. See Figure 2.

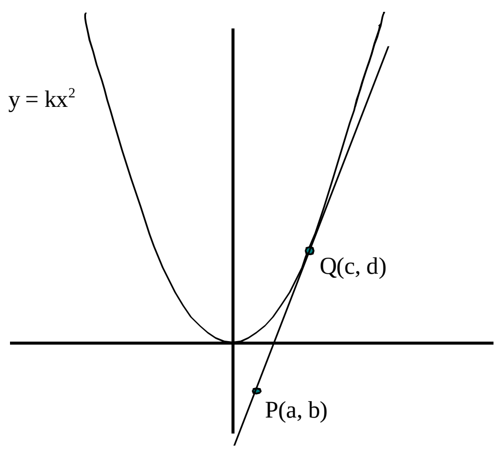

Figure 2

The problem becomes thus: Given point $P(a,b)$, find, if possible, point $Q(c,d)$ so that line $\overleftrightarrow{PQ}$ is tangent to the parabola at Q. The slope of line $\overleftrightarrow{PQ}$ has two representations:

(1) By the definition of slope, $m = \frac{d-b}{c-a}$.

(2) By the calculus, the slope of the line through Q, tangent to the parabola, is the derivative of the function $y = kx^2$, evaluated at $x = c$. Since $y' = 2kx$, the slope is $2kc$. Thus $m = 2kc$.

This gives us $m = \frac{d-b}{c-a} = 2kc$. Since $Q(c,d)$ lies on the parabola, the equation $d = kc^2$ holds. Substituting into the previous equation, we have $\frac{kc^2-b}{c-a} = 2kc$.

This yields:

$$
\begin{aligned}
& kc^2 - b = 2kc(c-a) \\
\Rightarrow\ & kc^2 - b = 2kc^2 - 2kca \\
\Rightarrow\ & 0 = kc^2 - (2ak)\,c + b
\end{aligned}
$$

(Call this equation the Critical Equation.)

To evaluate whether this quadratic equation has solutions for c, evaluate its discriminant.

$$
\begin{aligned}
D &= (-2ak)^2 - 4\,(k)\,(b) \\
&= 4a^2k^2 - 4kb \\
&= 4k\left(ka^2 - b\right)
\end{aligned}
$$

Two cases emerge:

(1) If $P(a,b)$ is in the interior of the parabola, then, by the inequality describing the interior, $b > ka^2$. Thus, we have $ka^2 - b < 0$ and $D < 0$. There are no real solutions to the critical equation. Hence no tangents to the parabola pass through point P.

(2) If $P(a,b)$ is in the exterior of the parabola, then, by the inequality describing the exterior, $b < ka^2$. Thus, we have $ka^2 - b > 0$ and $D > 0$. There are two solutions to the critical equation; therefore, two tangents to the parabola pass through point P.

Challenges to the Reader:

State and solve analogous problems for the ellipse and hyperbola.

Department of Mathematics
University of Northern Iowa
Cedar Falls, IA 50614-0506