Solutions and Discussions

Problem 1 — Volume 24, No. 1, Fall, 2000

Three circles are tangent to the same line at three distinct points. Also, each of these three circles is tangent to the other two circles. Two of the circles have radii with lengths 16 *and* 144. *Find all possible lengths for the radius of the third circle.*

Solution

Sheena Richards, Sophomore, Troy State University, Troy, AL

Case 1

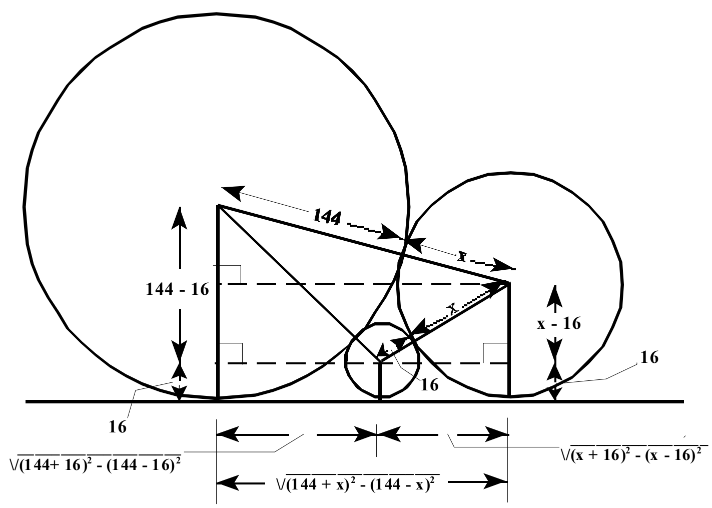

Figure 1

Utilizing the Pythagorean Theorem three times, with triangles having the marked right angles, yields the three distance calculations shown in Figure 1. This gives rise to the equation:

$$\sqrt{(144+16)^2-(144-16)^2}+\sqrt{(x+16)^2-(x-16)^2}=$$
$$\sqrt{(144+x)^2-(144-x)^2}$$

Squaring both sides, rearranging terms to isolate the remaining radical, and squaring a second time yields (after simplifying) the following quadratic equation:

$$\frac{64}{9}x^2-320x+3204=0$$

This has, as solutions, $x=9,36$. As $x>16$ in our diagram, we disregard $x=9$ and keep $x=36$.

Case 2

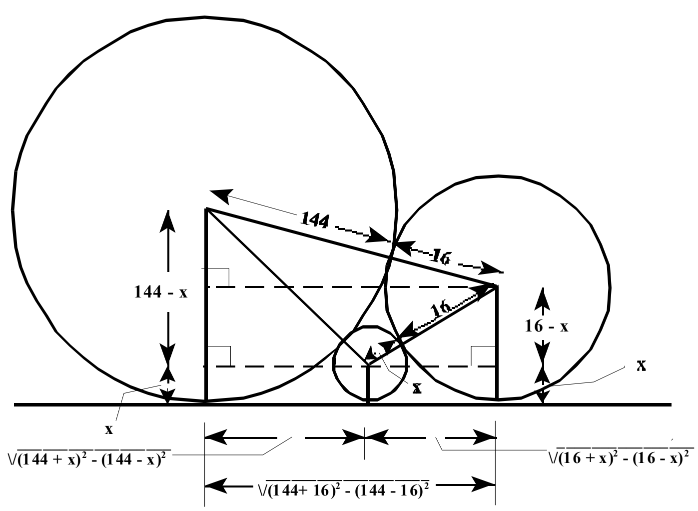

Figure 2

Following a nearly identical procedure, the same quadratic equation is produced. As $x<16$ in our diagram, we disregard $x=36$ and retain $x=9$.

Thus the possible radii are 9 and 36.

Problem 2 — Volume 24, No. 2, Fall, 2000.

Recall that $e^{ix} = \cos(x) + i\sin(x)$, *where* $i = \sqrt{-1}$ *denotes the imaginary number. Use this fact to prove that* i^i *has an infinite number of distinct values. Also, determine at least three of these values.*

Solution
Taoufiq Bellamine, Freshman, Troy State University.
Given $e^{ix} = \cos(x) + i\sin(x)$, we can isolate i on the right hand side of the equation by using $x = \frac{\pi}{2} + 2k\pi$, where k is any integer, since

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = 0 \qquad \text{and}$$
$$\sin\left(\frac{\pi}{2} + 2k\pi\right) = 0 \qquad \text{for } k \in \mathbf{Z}.$$

Thus, $i = e^{i\left(\frac{\pi}{2}+2k\pi\right)}$ for $k \in \mathbf{Z}$, and hence, $i^i = \left[e^{i\left(\frac{\pi}{2}+2k\pi\right)}\right]^i = e^{-\left(\frac{\pi}{2}+2k\pi\right)}$.

Since the real valued function, e^x, is one to one, and there are infinitely many values for k, there are infinitely many values for i^i.

Three specific values, using $k = 0, 1, 2$, respectively, are: $e^{-\frac{\pi}{2}}$, $e^{-\frac{5\pi}{2}}$, and $e^{-\frac{9\pi}{2}}$.

Problem 1 — Volume 25, No. 1, Spring, 2001.

Toss 3 *fair* 20 - *sided dice, one of which is red and the others blue. Determine the probability that the value shown on the red die lies between the values on the blue dice (inclusively).*

Editor's Comment: We include two solutions to the 20-sided dice problem, as they were done simultaneously and independently, and each is instructive in its own right. Note the two different approaches – one places the two blue dice first and fits the red die between them; the other places the red die first, and then places the blue dice around it. The latter is readily generalized to the corresponding n-sided dice problem.

First Solution:
Charles Kevin White, student, Auburn University at Montgomery

Let E be the event that the value shown on the red die lies between the values on the blue dice (inclusively). Mark one of the blue dice to distinguish between the two. Let a possible outcome be of the form: (blue die 1, blue die 2, red die). Since there are 20 possibilities for each die, there are a total of $20^3 = 8000$ possible outcomes.

If the two blue dice are rolled, then there can be, at most, 18 numbers strictly between the two. (If one of the two blue dice is 1 and the other is 20, then the numbers 2-19 are strictly between the two numbers on the blue dice.) The columns of the accompanying table display the interval (of spaces) between the two blue dice, the number of ways to have this interval between the two blue dice, and the number of successful outcomes resulting in this interval.

Interval (of Spaces) Between the Blue Dice	Number of Ways to Achieve This Interval	Number of Successful Outcomes Resulting in This Interval
18	$1 \cdot 2$	$2 \cdot 1 \cdot 20 = 40$
17	$2 \cdot 2$	$2 \cdot 2 \cdot 19 = 76$
16	$3 \cdot 2$	$2 \cdot 3 \cdot 18 = 108$
15	$4 \cdot 2$	$2 \cdot 4 \cdot 17 = 136$
14	$5 \cdot 2$	$2 \cdot 5 \cdot 16 = 160$
13	$6 \cdot 2$	$2 \cdot 6 \cdot 15 = 180$
12	$7 \cdot 2$	$2 \cdot 7 \cdot 14 = 196$
11	$8 \cdot 2$	$2 \cdot 8 \cdot 13 = 208$
10	$9 \cdot 2$	$2 \cdot 9 \cdot 12 = 216$
9	$10 \cdot 2$	$2 \cdot 10 \cdot 11 = 220$
8	$11 \cdot 2$	$2 \cdot 11 \cdot 10 = 220$
7	$12 \cdot 2$	$2 \cdot 12 \cdot 9 = 216$
6	$13 \cdot 2$	$2 \cdot 13 \cdot 8 = 208$
5	$14 \cdot 2$	$2 \cdot 14 \cdot 7 = 196$
4	$15 \cdot 2$	$2 \cdot 15 \cdot 6 = 180$
3	$16 \cdot 2$	$2 \cdot 16 \cdot 5 = 160$
2	$17 \cdot 2$	$2 \cdot 17 \cdot 4 = 136$
1	$18 \cdot 2$	$2 \cdot 18 \cdot 3 = 108$
0	$19 \cdot 2$	$2 \cdot 19 \cdot 2 = 76$
0	$20 \cdot 1$	$1 \cdot 20 \cdot 1 = 20$

Now why is the number of ways to have an interval of 18 spaces $1 \cdot 2$? This would be because in order to have an interval of 18 spaces, you must have one blue die having a 1 showing, and the other blue die having a 20 showing. It is not important which is which, $(1, 20, \textit{red die value})$ or $(20, 1, \textit{red die value})$. The reasoning is similar in all other cases, with the exception of the case in which both blue dice show the same number. In this case, there are 20 possible ways that the pair of blue dice can assume the same value.

The entries in the third column represent the number of ways to achieve a success, given a particular "interval" between the two blue dice. Each entry is computed by multiplying the number of values that the red die can assume which result in a success (i.e., the red die value being between the two blue dice values inclusive), by the number of ways that the corresponding "interval" between the two blue dice can be achieved.

The sum of the entries in the third column is 3060. This represents the total number of ways to achieve a success. If you divide this by the total number of possible outcomes, you get: $(3060)\,/\,(8000) = 0.3825$.

Therefore there is a 38.25% chance that the value shown on the red die lies between the values on the blue dice (inclusively).

Also solved by Taoufiq Bellamine, Freshman, Troy State University.

Second Solution:
Richard Harem, Senior, Troy State University

A generalized version of the problem is solved using 3 dice of n sides each. the blue dice are assumed to be distinguishable. The dice are labeled R, B_1, B_2. Equally likely outcomes are sequences of length 3 (in the order R, B_1, B_2, say) and there are n^3 of these. A successful outcome has the red die value between the blue dice values, inclusive. To count the number of successes, three cases are considered:

Case 1: $B_1 < R \leq B_2$
Case 2: $B_1 = R$
Case 3: $B_2 < R \leq B_1$

Case 1: Fix the value of R at k, where $1 \leq k \leq n$. There are $(k-1)$ choices for B_1. There are $(n-k+1)$ choices for B_2. By product of choices, there are $(k-1)(n-k+1)$ possibilities for a fixed k. There are $\sum_{k=1}^{n}(k-1)(n-k+1)$ successes satisfying $B_1 < R \leq B_2$.

Case 2: Fix the value of R at k, where $1 \leq k \leq n$. There is only one choice for B_1, namely k. There are n choices for B_2.There are n possibilities for a fixed k. Thus, there are $\sum_{k=1}^{n} n$ successes satisfying $B_1 = R$.

Case 3: Fix the value of R at k, where $1 \leq k \leq n$. There are $(n-k)$ choices for B_1. There are k choices for B_2. By product of choices, there are $k\,(n-k)$ possibilities for a fixed k. There are $\sum_{k=1}^{n} k\,(n-k)$ successes satisfying $B_2 \leq R < B_1$.

Combining, there are

$$\sum_{k=1}^{n} [(k-1)(n-k+1) + n + k(n-k)]$$

ways of achieving a success. Rearranging terms, this yields:

$$\sum_{k=1}^{n} \left[-2k^2 + (2n+2)k - 1\right] =$$

$$-2\sum_{k=1}^{n} k^2 + (2n+2)\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1.$$

Using well-known summation formulas, the first term can be re-written:

$$-2\frac{n(n+1)(2n+1)}{6} = \frac{-2n^3 - 3n^2 - n}{3}.$$

The second term can be re-written:

$$(2n+2)\frac{n(n+1)}{2} = \frac{3n^3 + 6n^2 + 3n}{3}.$$

The third term is $-n = -\frac{3n}{3}$.

Combining the three terms yields, $\frac{n^3+3n^2-n}{3}$, the total number of possible successes. Dividing by n^3 produces $\frac{n^2+3n-1}{3n^2}$ as the probability of success for the n-sided dice problem. In particular, for $n = 20$, the probability of success is $\frac{459}{1200} = .3825 = 38.25\%$.

Problem 3 — Volume 25, No. 1, Spring, 2001

An urn contains one red marble and one blue marble. Draw two marbles with replacement. You win if both marbles are red. Otherwise, add one more blue marble and repeat. (That is, again draw two marbles with replacement.) Continue repeating this process until both marbles are red. determine the probability that you will eventually win.

Solution

Bruce Myers, Kankakee Community College, Kankakee, IL.

Beginning with one red marble and one blue marble, the probability of winning the game on the first attempt is $\frac{1}{4}$ ($\frac{1}{2}$ chance of drawing a red marble on each turn, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$).

In order to win on the second attempt, the player must lose on the first attempt (probability of $\frac{3}{4}$), and then be successful on the second attempt (probability $\frac{1}{9}$, as there are now two blue marbles and one red marble in the urn). Therefore, the probability of winning on the second attempt is $\frac{1}{12}$ ($\frac{3}{4} \cdot \frac{1}{9}$).

In order to win on the third attempt, the player must lose on attempt 1 (probability of $\frac{3}{4}$), lose on attempt 2 (probability $\frac{8}{9}$), and win on attempt 3 (probability $\frac{1}{16} = \frac{1}{4} \cdot \frac{1}{4}$). Therefore, the probability of winning on the third attempt is $\frac{1}{24} = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16}$.

In general, the probability of winning on the n^{th} attempt is given by

$$\left(\frac{2^2-1}{2^2} \cdot \frac{3^2-1}{3^2} \cdot \ldots \cdot \frac{n^2-1}{n^2} \right) \cdot \frac{1}{(n+1)^2} \quad \text{or}$$

$$\left(\prod_{k=2}^{n} \frac{k^2-1}{k^2} \right) \cdot \frac{1}{(n+1)^2}$$

A pattern is sought to simplify this expression. The probabilities of winning in the first six attempts are listed in the following chart:

Attempt	Probability of Winning
1	$\frac{1}{4}$
2	$\frac{1}{12}$
3	$\frac{1}{24}$
4	$\frac{1}{40}$
5	$\frac{1}{60}$
6	$\frac{1}{84}$

Consider the sequence of denominators: $4, 12, 24, 40, 60, 84, \ldots$. The sequence of differences of consecutive values is $8, 12, 16, 20, 24, \ldots$, and is arithmetic with common difference of 4. Using a method described in most texts of Finite Mathematics, the n^{th} term of the original sequence of denominators is given by $2n + 2n^2$, or $2\left(n + n^2\right)$. It can be verified by induction that the probability of winning on the n^{th} attempt, already derived as $\left(\prod_{k=2}^{n} \frac{k^2-1}{k^2} \right) \cdot \frac{1}{(n+1)^2}$, is equal to $\frac{1}{2(n+n^2)}$ for all $n \in \mathbf{N}$.

The probability of ever winning the game is the sum of the probabilities for each individual attempt:

$$\sum_{n=1}^{\infty} \frac{1}{2\left(n+n^2\right)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\left(n+n^2\right)} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

The n^{th} partial sum of the series can be written as:

$$\frac{1}{2} \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \right].$$

By removing parenthesis and combining terms, the expression reduces (in "spyglass fashion") to: $\frac{1}{2}\left[1 - \frac{1}{n+1}\right]$. The limit of the n^{th} partial sum, as $n \to \infty$, is $\frac{1}{2}$. Therefore, the probability of eventually winning the game, as defined, is

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(n+n^2)}=\frac{1}{2}\,.$$

Problem 6 — Volume 25, No. 1, Spring, 2001

Solve this system of equations, where x, y are positive, and a, b are any real numbers.

$$\begin{array}{rcrcl} x^2 & + & y^2 & = & a^2 \\ \ln(x) & + & \ln(y) & = & b \end{array}$$

[This problem is from *The Analyst,* vol. 9 (1882), pp. 32, 62.]

Solution
Taoufiq Bellamine, Freshman, Troy State University.

Doubling the second equation and simplifying with the properties of logarithms yields

$$\ln\left(x^2y^2\right)=2b \qquad \text{or, equivalently,}$$

$$x^2y^2=e^{2b}. \tag{1}$$

Solving for x^2 in the first equation of the original system, and substituting into Eq. 1 yields:

$$\left(a^2-y^2\right)y^2=e^{2b} \qquad \text{or, equivalently,}$$

$$y^4-a^2y^2+e^{2b}=0. \tag{2}$$

Eq. 2 is a quadratic equation in the unknown, y^2. The quadratic formula yields:

$$y^2=\frac{a^2\pm\sqrt{a^4-4e^{2b}}}{2} \qquad \text{and hence,}$$

$$y=\pm\sqrt{\frac{a^2\pm\sqrt{a^4-4e^{2b}}}{2}}$$

In order for y to be a real number, we must have $a^4-4e^{2b}\geq 0$ (Restriction 1). When this occurs, $a^2-\sqrt{a^4-4e^{2b}}\geq 0$ as well (Restriction 2).

Restriction 1 imposes the condition, after some simplification, $b\leq\ln\left(a^2\right)-\ln(2)\,.$

Given that this restriction is in effect, Restriction 2 is automatically satisfied, as $a^4\geq a^4-4e^{2b}\geq 0$, so extracting roots, yields $a^2\geq\sqrt{a^4-4e^{2b}}\geq 0$ as desired.

Since it is required that $y > 0$, we only include the positive root (when our restriction is in effect). So,

$$y_1 = \sqrt{\frac{a^2 + \sqrt{a^2 - 4e^{2b}}}{2}} \ ; \quad y_2 = \sqrt{\frac{a^2 - \sqrt{a^2 - 4e^{2b}}}{2}}$$

provided that $b \leq \ln\left(a^2\right) - \ln(2)$.

To find the corresponding x values, recall that $x^2 = a^2 - y^2$, so $x = \sqrt{a^2 - y^2}$ (as our original restrictions require that $x > 0$).

Thus, when $y = y_1$, $x = \sqrt{\frac{a^2-\sqrt{a^2-4e^{2b}}}{2}}$, and when $y = y_2$, $x = \sqrt{\frac{a^2+\sqrt{a^2-4e^{2b}}}{2}}$.

Our solution set is:

$$\left\{\left(\sqrt{\frac{a^2-\sqrt{a^2-4e^{2b}}}{2}}, \sqrt{\frac{a^2+\sqrt{a^2-4e^{2b}}}{2}}\right), \left(\sqrt{\frac{a^2+\sqrt{a^2-4e^{2b}}}{2}}, \sqrt{\frac{a^2-\sqrt{a^2-4e^{2b}}}{2}}\right)\right\}$$