Why a Collection of More Than n Vectors in $\mathbf{R}^n$ is Always Linearly Dependent

By Pat Rossi

1. Introduction

One concept that an instructor of Linear Algebra would like his or her students to grasp by the end of the course is that of linear independence. A fact, relating to linear independence, that a fair number of students eventually come to accept without really understanding, is that any collection of more than n vectors in $\mathbf{R}^n$ is linearly dependent.

In this paper, we consider an elementary proof of this fact, based on two fairly obvious lemmas. The value of this proof is that it can be presented in a less formal, more visual and intuitive way, that is easily understood by most students.

After presenting a rigorous version of this proof of why any collection of $n+1$ vectors in $\mathbf{R}^n$ is always linearly dependent, we will consider a less formal version more suited for the typical undergraduate student.

2. The Proof

To begin, we will agree to restrict allowable row operations, to be used during forward elimination, to the following three:

(1) Any row may be multiplied by a non-zero constant.
(2) Any row may be replaced by the sum of itself and a non zero multiple of another row.
(3) Any two rows may be "switched," or interchanged, position-wise.

Lemma 1. *Let A be an $n \times m$ matrix, and let $v_1, v_2, \ldots, v_n$ be the original rows of A. At any stage during forward elimination, every row of A is a linear combination of the original rows, $v_1, v_2, \ldots, v_n$.*

Proof. Let N be the number of "allowable" row operations that have been performed. We define $Row\,(i)$ to be the i^{th} row of the matrix after N row operations. The proof is by induction on N.

For $N = 0$, the lemma holds, as $row\,(i) = \sum_{j=1}^{m} \delta_{ij} v_j$.

For $N = k$, suppose that each row is a linear combination of the original rows, $v_1, v_2, \ldots, v_n$. Then for $i = 1, 2, \ldots, m$; $row\,(i) = \sum_{j=1}^{m} c_{ij} v_j$. For the induction step, we will consider the three cases separately.

(1) If the $k+1$ row operation is that of multiplying $row\,(p)$ by a non-zero constant c, then $row\,(p) = \sum_{j=1}^{m} (c \cdot c_{pj})\, v_j$.
(2) If the $k+1$ row operation is that of replacing $row\,(p)$ with the sum of $row\,(p) + c \cdot row\,(q)$, then $row\,(p) = \sum_{j=1}^{m} (c_{pj} + c \cdot c_{qj})\, v_j$.
(3) If the $k+1$ row operation is that of interchanging $row\,(p)$ and $row\,(q)$, then $row\,(p) = \sum_{j=1}^{m} c_{qj} v_j$ and $row\,(q) = \sum_{j=1}^{m} c_{pj} v_j$.

In each case, the other rows remain unchanged, and therefore, after $k+1$ row operations, each row is still a linear combination of the original rows $v_1, v_2, \ldots, v_n$. □

Lemma 2. *Let A be an $n \times m$ matrix, with $n > m$. Then forward elimination always yields a zero row.*

Proof. (By contradiction) Assume that row interchange is performed wherever necessary, so that the matrix is in Upper Echelon form. Suppose, for the sake of contradiction, that after forward elimination has been performed, there is no zero row. Let j be least, such that the j^{th} column contains a non-zero entry below the main diagonal. (Such an entry exists, otherwise row $m+1$ would be a zero row after elimination, contrary to our hypothesis.) For $i > j$, let i be least, such that $a_{ij} \neq 0$. Then $a_{jj} = 0$. Otherwise, a_{ij} would have been eliminated during forward elimination. This implies that the matrix is not in reduced echelon form, contradicting our earlier assumption. □

Theorem 1. *Any collection of more than n vectors in $\mathbf{R}^n$ is linearly dependent.*

Proof. Let $\{v_1, v_2, \ldots, v_n, \ldots, v_p\}$ be any collection of p vectors in $\mathbf{R}^n$, with $p > n$. Form a matrix, A, such that for $i = 1, 2, \ldots, p$, the i^{th} row of A is v_i. By Lemma 2, forward elimination will yield a zero row — we'll call it $row\,(i)$. By Lemma 1, $row\,(i)$ is a non-trivial linear combination of vectors $v_1, v_2, \ldots, v_n, \ldots, v_p$. That is:

$$row\,(i) = \sum_{j=1}^{p} c_{ij} v_j = 0\ ; \qquad \text{with not all } c_{ij} = 0.$$

□

One of the nice things about this proof, is that it can be presented in a less formal, more visual way, which is easily understood by most students in a first undergraduate linear algebra class. Through experience, students realize intuitively, that any non-zero entry below the main diagonal can be eliminated using the three allowable row operations. What is more difficult to comprehend, is that during any stage of forward elimination, each row is a linear combination of the original rows. It is helpful in this regard, to introduce the concept of linear combination before students have occasion to perform elimination on matrices. Then, when elimination, using the three allowable row operations is introduced, repetitive emphasis can be placed on the fact that any row altered by such a sequence of operations is a linear combination of the original rows. Having this understanding, the students are well-equipped to understand why any collection of $n+1$ vectors in $\mathbf{R}^n$ is linearly dependent.

3. The Presentation

Given a collection $\{v_1, v_2, \ldots, v_n, \ldots, v_p\}$ of more than n vectors in $\mathbf{R}^n$, form a matrix, A, whose first row is v_1, second row is v_2, etc. The construction is illustrated in Figure 1.

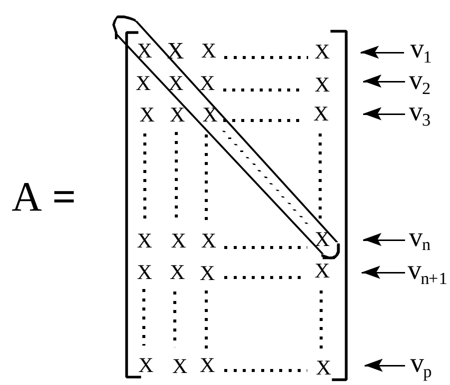

Figure 1

Note that A has more rows than columns. Performing forward elimination, and keeping the matrix in upper echelon form, we obtain zero rows below the main diagonal. The situation is illustrated in Figure 2. (To avoid needless confusion, mention that this does not necessarily mean that all entries on or above the main diagonal are non-zero.)

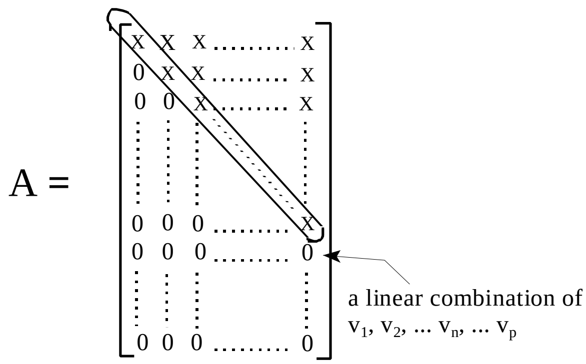

Figure 2

Since each zero row is the result of forward elimination, and hence, a non-trivial linear combination of $v_1, v_2, \ldots, v_n, \ldots, v_p$, the collection of vectors, $\{v_1, v_2, \ldots, v_n, \ldots, v_p\}$, is linearly dependent.

Department of Math & Physics
Troy State University
Troy, AL 36082
prossi@trojan.troyst.edu