Introduction to Analytic Fibonometry

By Robyn Minor Smith

1. Introduction

It is well known that the differential equations

$$y'' + y = 0, \text{ with } y(0) = 0 \text{ and } y'(0) = 1$$

and

$$y'' - y = 0, \text{ with } y(0) = 0 \text{ and } y'(0) = 1$$

lead to circular trigonometry and hyperbolic trigonometry respectively. On the other hand, the initial value problem,

$$y'' - y' - y = 0, \text{with } y(0) = 0 \text{ and } y'(0) = 1 \qquad (1)$$

is the differential equation analogue to the well-known recursion equation $c_n = c_{n-1} + c_{n-2}$, with $c_0 = 0$ and $c_1 = 1$. This leads us quite naturally to the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$; and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. We will define the initial value problem (Eq. 1) to be the *Fibonacci Differential Equation* (FDE). It has been suggested that there is a kind of trigonometry that can be associated with certain kinds of differential equations. This paper investigates some of the topics associated with the trigonometry derived from the Fibonacci Differential Equation. The functions we will develop will be defined as the *Fibonometric functions.*

2. The Fibonometric Sine and Cosine

It is straight forward to show that the solution to the FDE is $y = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Notice that α and β are solutions of the quadratic equation $s^2 - s - 1 = 0$. Following the well-known patterns of $sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ and $\sinh(x) = \frac{e^{x} - e^{-x}}{2}$, we define the solution of the FDE (1):

DEFINITION. *The* ***Fibonacci sine*** *is*

$$sinf(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$.

A series solution can also be developed. We know that $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution of the FDE. We take the first and second derivatives of this and substitute into the FDE. After combining like terms and equating like coefficients in the usual way, we obtain the recursion relation

$$(n+1)(n+2)\,c_{n+2} - (n+1)\,c_{n+1} - c_n = 0.$$

We use this equation to prove the following:

LEMMA. $c_n = \frac{F_n c_1 + F_{n-1} c_0}{n!}$, *where* F_n *is the* n^{th} *Fibonacci number.*

PROOF. We use strong induction.

Basis: $c_2 = \frac{F_2 c_1 + F_1 c_0}{2!} = \frac{(1)c_1 + (1)c_0}{2!}$

Assumption: $c_{n+1} = \frac{F_{n+1} c_1 + F_n c_0}{(n+1)!}$ for all $n+1 \geq 3$

Induction:

$$(n+1)(n+2)c_{n+2} - (n+1)c_{n+1} - c_n = 0,$$
$$(n+1)(n+2)c_{n+2} = (n+1)c_{n+1} + c_n,$$
$$(n+1)(n+2)c_{n+2} = (n+1)\left(\frac{F_{n+1}c_1 + F_n c_0}{(n+1)!}\right) + \left(\frac{F_n c_1 + F_{n-1} c_0}{n!}\right),$$
$$(n+1)(n+2)c_{n+2} = \frac{F_{n+1}c_1 + F_n c_0}{n!} + \frac{F_n c_1 + F_{n-1} c_0}{n!},$$
$$c_{n+2} = \frac{(F_{n+1} + F_n)c_1 + (F_n + F_{n-1})c_0}{(n+2)(n+1)n!},$$
$$c_{n+2} = \frac{F_{n+2}c_1 + F_{n+1}c_0}{(n+2)!}$$

□

Now we can re-express the solution of the differential equation

$$y = c_0 + c_1 x + \left(\frac{c_1 + c_0}{2!}\right)x^2 + \cdots + \left(\frac{F_n c_1 + F_{n-1} c_0}{n!}\right)x^n + \cdots.$$

Imposing the initial conditions $y(0) = 0$ and $y'(0) = 1$ on the series, we have $c_0 = 0$ and $c_1 = 1$. Therefore,

$$\begin{aligned} y &= (1)x + \left(\tfrac{1}{2!}\right)x^2 + \left(\tfrac{2}{3!}\right)x^3 + \left(\tfrac{3}{4!}\right)x^4 + \cdots + \left(\tfrac{F_n}{n!}\right)x^n + \cdots \\ &= \sum_{n=0}^{\infty} F_n \frac{x^n}{n!}. \end{aligned}$$

Therefore, we have proved

THEOREM 1. *The Fibonacci sine is*

$$sinf(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_n \frac{x^n}{n!},$$

where F_n *is the* n^{th} *Fibonacci number.*

The convergence of the series can be determined by the ratio test. In the ratio test the quotient of the $n+1$ and the n coefficients of the series is examined as $n \to \infty$:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{F_{n+1}}{(n+1)!}}{\frac{F_n}{n!}} \right| &= \lim_{n\to\infty} \left[\left(\frac{F_{n+1}}{(n+1)!} \right) \left(\frac{n!}{F_n} \right) \right], \\
&= \lim_{n\to\infty} \left[\left(\frac{n!}{(n+1)!} \right) \left(\frac{F_{n+1}}{F_n} \right) \right], \\
&= \lim_{n\to\infty} \left[\left(\frac{1}{n+1} \right) \left(\frac{F_{n+1}}{F_n} \right) \right], \\
&= 0 \cdot \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n} \right).
\end{aligned}$$

In order to determine the limit, $\lim\limits_{n\to\infty} \left(\frac{F_{n+1}}{F_n} \right)$, the Binet Form of the Fibonacci numbers will be used. The Binet Form of the n^{th} Fibonacci number, F_n is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{2}$$

Note that this formula can be readily obtained from the solution of the difference equation $c_{n+1} = c_n + c_{n-1}$. With this characterization of the Fibonacci numbers, the ratio test for convergence of $sinf(x)$ can be completed. So

$$\begin{aligned}
0 \cdot \left(\frac{F_{n+1}}{F_n} \right) &= 0 \cdot \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) \left(\frac{\alpha - \beta}{\alpha^n - \beta^n} \right), \\
&= 0 \cdot \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} \right), \\
&= 0 \cdot \frac{\lim\limits_{n\to\infty} \alpha^{n+1} - \lim\limits_{n\to\infty} \beta^{n+1}}{\lim\limits_{n\to\infty} \alpha^n - \lim\limits_{n\to\infty} \beta^n}.
\end{aligned}$$

Notice, $|\beta| < 1$. Therefore, $\beta^n \to 0$ as $n \to \infty$. Hence,

$$0 \cdot \left(\frac{\lim\limits_{n\to\infty} \alpha^{n+1}}{\lim\limits_{n\to\infty} \alpha^n} \right) = 0 \cdot \lim_{n\to\infty} \alpha = 0 \cdot \alpha = 0.$$

Therefore the series expansion for $sinf(x)$ is absolutely convergent for all real numbers x.

We define $cosf(x) = \frac{d}{dx}(sinf(x))$, or equivalently, from Theorem 1:

DEFINITION. *The **Fibonacci cosine** is*

$$cosf(x) = \frac{\alpha e^{\alpha x} - \beta e^{\beta x}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_{n+1} \frac{x^n}{n!},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$, *respectively.*

LEMMA. $cosf(x) \neq 0$ *for all* x.

PROOF. Suppose that $cosf(x) = 0$ for some x, then we would have

$$cosf(x) = \frac{\alpha e^{\alpha x} - \beta e^{\beta x}}{\alpha - \beta} = 0,$$

or

$$\begin{aligned} \alpha e^{\alpha x} - \beta e^{\beta x} &= 0, \\ \alpha e^{\alpha x} &= \beta e^{\beta x}, \\ \frac{\beta}{\alpha} &= e^{(\alpha - \beta)}. \end{aligned}$$

But

$$\frac{\beta}{\alpha} = \frac{1-\sqrt{5}}{1+\sqrt{5}} = \frac{(1-\sqrt{5})^2}{1-5} = \frac{(1-\sqrt{5})^2}{-4} = -\left(\frac{1-\sqrt{5}}{2}\right)^2 = -\beta^2.$$

Since e raised to any real power is positive and $\frac{\beta}{\alpha} < 0$, we have a contradiction and $cosf(x) \neq 0$. □

The convergence of $cosf(x)$ is guaranteed since the derivative of an absolutely convergent series is convergent. Since the remainder of the Fibonacci functions are defined in terms of quotients of either $sinf(x)$ or $cosf(x)$, they will also be absolutely convergent except when a denominator is 0.

3. The Fibonometric Tangent and Cotangent

In this section we will follow the familiar patterns for the circular and hyperbolic functions and define the analogous Fibonometric functions. We define $tanf(x) = \frac{sinf(x)}{cosf(x)}$. As we have shown, this function is defined for all real numbers x.

Since the power series expansions for $sinf$ and $cosf$ have particularly nice coefficients with respect to the Fibonacci numbers, we investigate two types of series expansions for the other Fibonometric functions. The first expansion is in terms of powers of e^x.

By dividing the Fibonacci sine, $\frac{e^{\alpha x}-e^{\beta x}}{\alpha-\beta}$, by the Fibonacci cosine, $\frac{\alpha e^{\alpha x}-\beta e^{\beta x}}{\alpha-\beta}$, we have

$$\begin{aligned} tanf(x) &= \frac{e^{\alpha x}-e^{\beta x}}{\alpha e^{\alpha x}-\beta e^{\beta x}}, \\ &= \frac{e^{\alpha x}}{e^{\alpha x}} \cdot \left(\frac{1-e^{\beta x-\alpha x}}{\alpha-\beta e^{\beta x-\alpha x}} \right), \\ &= \frac{\alpha}{\alpha} \cdot \left(\frac{\frac{1}{\alpha}-\frac{1}{\alpha}e^{(\beta-\alpha)x}}{1-\frac{\beta}{\alpha}e^{(\beta-\alpha)x}} \right). \end{aligned}$$

Note:

$$\begin{aligned} \tfrac{1}{\alpha} &= \tfrac{2}{1+\sqrt{5}} = \tfrac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} \\ &= -\tfrac{1-\sqrt{5}}{2} = -\beta; && (3) \\ \alpha-\beta &= \tfrac{1+\sqrt{5}}{2} - \tfrac{1-\sqrt{5}}{2} = \tfrac{2\sqrt{5}}{2} = \sqrt{5}; && (4) \\ \tfrac{\beta}{\alpha} &= -\beta^2. && (5) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\frac{1}{\alpha}-\frac{1}{\alpha}e^{(\beta-\alpha)x}}{1-\frac{\beta}{\alpha}e^{(\beta-\alpha)x}} &= \frac{-\beta+\beta e^{-\sqrt{5}x}}{1+\beta^2 e^{-\sqrt{5}x}}, \\ &= \beta\left(\frac{-1+e^{-\sqrt{5}x}}{1+\beta^2 e^{-\sqrt{5}x}} \right). \end{aligned}$$

We perform the indicated division, and use the fact that β is a solution of the equation $s^2-s-1=0$ to replace β^2+1 with $\beta+2$, to obtain:

$$\begin{aligned} \beta\left(\frac{-1+e^{-\sqrt{5}x}}{1+\beta^2 e^{-\sqrt{5}x}} \right) &= \beta\Big[-1+(\beta+2)e^{-\sqrt{5}x} - \beta^2(\beta+2)e^{-2\sqrt{5}x} \\ &\quad +\beta^4(\beta+2)e^{-3\sqrt{5}x} - \cdots \Big]. \end{aligned}$$

This yields

$$\begin{aligned}
tanf(x) &= \beta\left[-1+\sum_{n=0}^{\infty}(-1)^n\beta^{2n}(\beta+2)e^{-(n+1)\sqrt{5}x}\right].\\
&= \beta\left[-1+(\beta+2)\sum_{n=0}^{\infty}(-1)^n\beta^{2n}e^{-(n+1)\sqrt{5}x}\right],\\
&= -\beta+\beta(\beta+2)\sum_{n=0}^{\infty}(-1)^n\beta^{2n}e^{-(n+1)\sqrt{5}x},\\
&= -\beta+(\beta^2+2\beta)\sum_{n=0}^{\infty}(-1)^n\beta^{2n}e^{-(n+1)\sqrt{5}x},\\
&= -\beta+(\beta+1+2\beta)\sum_{n=0}^{\infty}(-1)^n\beta^{2n}e^{-(n+1)\sqrt{5}x},\\
&= -\beta+(3\beta+1)\sum_{n=0}^{\infty}(-1)^n\beta^{2n}e^{-(n+1)\sqrt{5}x}.
\end{aligned}$$

Therefore, β^{2n} are coefficients of powers of $e^{-\sqrt{5}x}$. However, notice:

$$\begin{aligned}
\beta^2 &= 1+\beta;\\
\beta^3 &= \beta(1+\beta)=\beta+\beta^2=1+2\beta;\\
\beta^4 &= \beta(1+2\beta)=\beta+2\beta^2=\beta+2(1+\beta)=2+3\beta;\\
\beta^5 &= \beta(2+3\beta)=2\beta+3\beta^2=2\beta+3(1+\beta)=3+5\beta,
\end{aligned}$$

which suggests the following proposition.

PROPOSITION. *$\beta^n = F_{n-1}+F_n\beta$ for $n \geq 1$ and similarly, $\alpha^n = F_{n-1}+F_n\alpha$ for $n \geq 1$.*

PROOF. We will only give the proof for β^n using induction.

$$\begin{aligned}
&\text{Basis:} & \beta^2 &= F_1+F_2\beta.\\
&\text{Assumption:} & \beta^n &= F_{n-1}+F_n\beta.\\
&\text{Induction:} & \beta^{n+1} &= \beta(\beta^n)=\beta(F_{n-1}+F_n\beta),\\
&& &= F_{n-1}\beta+F_n\beta^2,\\
&& &= F_{n-1}\beta+F_n(\beta+1),\\
&& &= F_{n-1}\beta+F_n\beta+F_n,\\
&& &= (F_{n-1}+F_n)\beta+F_n,\\
&& &= F_{n+1}\beta+F_n.
\end{aligned}$$

□

Hence we have proved

THEOREM 2. *The Fibonacci tangent can be expressed*

$$\begin{aligned} tanf(x) &= -\beta + (3\beta+1)\Bigg[e^{-\sqrt{5}x} \\ &\quad + \sum_{n=1}^{\infty}(-1)^n(F_{n-1}+F_n\beta)^2 e^{-\sqrt{5}(n+1)x}\Bigg] \end{aligned}$$

where $\beta = \frac{1-\sqrt{5}}{2}$.

As a bonus, we note that the Binet form of the n^{th} Fibonacci number, equation 2, can be derived from the Proposition

$$\begin{aligned} \beta^n &= F_{n-1} + F_n\beta, \\ \alpha^n &= F_{n-1} + F_n\alpha, \\ \alpha^n - \beta^n &= F_n(\alpha-\beta), \\ F_n &= \frac{\alpha^n - \beta^n}{\alpha-\beta}. \end{aligned}$$

While the coefficients in the series for $tanf(x)$ in powers of e^x were of some interest, the coefficients of the series in powers of x turned out to be very complicated. We were not able to obtain a closed form general term.

The Fibonacci $cotf(x)$ can be defined in the obvious way

$$cotf(x) = \frac{cosf(x)}{sinf(x)} = \frac{\alpha e^{\alpha x} - \beta e^{\beta x}}{e^{\alpha x} - e^{\beta x}}.$$

We investigate the involvement of the Fibonacci numbers in the coefficients of this quotient. Recalling equations 3, 4, and 5, we have

$$\begin{aligned} cotf(x) &= \left(\frac{\alpha}{\alpha}\right)\left[\frac{e^{\alpha x} - \left(\frac{\beta}{\alpha}\right)e^{\beta x}}{\left(\frac{1}{\alpha}\right)e^{\alpha x} - \left(\frac{1}{\alpha}\right)e^{\beta x}}\right], \\ &= \left(\frac{e^{\alpha x}}{e^{\alpha x}}\right)\left[\frac{1+\beta^2 e^{-\sqrt{5}x}}{-\beta+\beta e^{-\sqrt{5}x}}\right], \\ &= \left(\frac{1}{\beta}\right)\left[\frac{1+(\beta+1)\,e^{-\sqrt{5}x}}{-1+e^{-\sqrt{5}x}}\right]. \end{aligned}$$

Carrying out the division in the second factor, we obtain

$$\frac{1+(\beta+1)\,e^{-\sqrt{5}x}}{-1+e^{-\sqrt{5}x}} = -1 - (\beta+2)e^{-\sqrt{5}x} - (\beta+2)e^{-2\sqrt{5}x} - \cdots.$$

Consequently, we have

$$cotf(x) = -\frac{1}{\beta}\left(1 + (\beta+2)\sum_{n=1}^{\infty} e^{-n\sqrt{5}x}\right).$$

Note, this is undefined if $x = 0$, for then $sinf(x) = 0$, i.e. the series fails to converge since

$$(\beta + 2) \sum_{n=1}^{\infty} 1 = (\beta + 2) \lim_{n \to \infty} n = \infty.$$

Furthermore, for any $x < 0$, each of the terms in the series for $cotf(x)$ is a positive power of e, and hence, is greater than 1. Thus, this series diverges for $x \leq 0$; or, to state it positively, the series expression for $cotf(x)$, derived above, converges for $x > 0$. As with the Fibonacci tangent, a generalized expression for the coefficients in the power series in x is not readily obtainable.

Definitions for the Fibonacci secant and cosecant can be obtained analogously and their series investigated. The general terms are not readily obtainable.

4. Elementary Identities of Fibonometry

In circular trigonometry, one has the identity $\sin^2 t + \cos^2 t = 1$. To eliminate the parameter t, one lets $x = \cos(t)$ and $y = \sin(t)$ and obtains the circle $x^2 + y^2 = 1$. In hyperbolic trigonometry, the identity $\cosh^2 t - \sinh^2 t = 1$ leads to hyperbola $x^2 - y^2 = 1$, where $x = \cosh t$ and $y = \sinh t$. However, the situation is not so direct in Fibonometry. The proof of the following theorem is straightforward but tedious. the motivation for the theorem is not so clear and would take us too far afield for this paper.

Theorem 3. *The Fundamental Identity of Fibonometry is*

$$cosf^2(x) - cosf(x)sinf(x) - sinf^2(x) = e^x.$$

Proof. Recall that $s^2 - s - 1 = 0$ for $s = \alpha$ or β. Therefore

$$\begin{aligned}
& \left(\tfrac{\alpha e^{\alpha x} - \beta e^{\beta x}}{\alpha - \beta}\right)^2 - \left(\tfrac{\alpha e^{\alpha x} - \beta e^{\beta x}}{\alpha - \beta}\right)\left(\tfrac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}\right) \\
& - \left(\tfrac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}\right)^2 \\
= \; & \left(\tfrac{\alpha^2 e^{2\alpha x} - 2\alpha\beta e^{(\alpha+\beta)x} + \beta^2 e^{2\beta x}}{\alpha^2 - 2\alpha\beta + \beta^2}\right) \\
& - \left(\tfrac{\alpha e^{2\alpha x} - \alpha e^{(\alpha+\beta)x} - \beta e^{(\alpha+\beta)x} + \beta e^{2\beta x}}{\alpha^2 - 2\alpha\beta + \beta^2}\right) \\
& - \left(\tfrac{e^{2\alpha x} - 2e^{(\alpha+\beta)x} + e^{2\beta x}}{\alpha^2 - 2\alpha\beta + \beta^2}\right), \\
= \; & \tfrac{\left(\alpha^2 - \alpha - 1\right)e^{2\alpha x}}{\alpha^2 - 2\alpha\beta + \beta^2} + \tfrac{(\beta - 2\alpha\beta + \alpha + 2)e^{(\alpha+\beta)x}}{\alpha^2 - 2\alpha\beta + \beta^2} \\
& + \tfrac{\left(\beta^2 - \beta - 1\right)e^{2\beta x}}{\alpha^2 - 2\alpha\beta + \beta^2}, \\
= \; & \left(\tfrac{0 \cdot e^{2\alpha x} + 5e^x + 0 \cdot e^{2\beta x}}{5}\right) = e^x.
\end{aligned}$$

□

Note that the proof of the last theorem depends on the value of α and β only at the last step. This leads to the following generalization, which will play an important role in the second part of this work.

THEOREM 4. *Suppose p and q are distinct real numbers or conjugate complex numbers. If*

$$y(x) = \frac{(e^{px} - e^{qx})}{p-q}$$

for all real numbers x, then

$$(y')^2 - (p+q)\,yy' + pqy^2 = e^{(p+q)x}$$

and

$$(y' - py)^p = (y' - qy)^q = e^{pqx}.$$

PROOF. For the given function,

$$\begin{aligned}
&(y')^2 - (p+q)\,yy' + pqy^2 \\
=\;& \left(\tfrac{pe^{px} - qe^{qx}}{p-q}\right)^2 - (p+q)\left(\tfrac{e^{px} - e^{qx}}{p-q}\right)\left(\tfrac{pe^{px} - qe^{qx}}{p-q}\right) \\
&+ pq\left(\tfrac{e^{px} - e^{qx}}{p-q}\right)^2, \\
=\;& \tfrac{\left(p^2 e^{(p+q)x} - 2pqe^{(p+q)x} + q^2 e^{(p+q)x}\right)}{(p-q)^2}, \\
=\;& \tfrac{\left(p^2 - 2pq + q^2\right)e^{(p+q)x}}{(p-q)^2}, \\
=\;& \tfrac{(p-q)^2 e^{(p+q)x}}{(p-q)^2} = e^{(p+q)x}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(y' - py)^p &= \left[\left(\tfrac{pe^{px} - qe^{qx}}{p-q}\right) - p\left(\tfrac{(e^{px} - e^{qx})}{p-q}\right)\right]^p, \\
&= \left(\tfrac{pe^{px} - qe^{qx} - pe^{px} + pe^{qx}}{p-q}\right)^p, \\
&= \left(\tfrac{(p-q)e^{qx}}{p-q}\right)^p = e^{pqx}.
\end{aligned}$$

The proof is similar for

$$(y' - qy)^q = e^{pqx}.$$

□

5. Conclusions

In the previous sections we have defined the basic Fibonometric functions and shown their series expansions. In addition, we have suggested how these may be similar to the trigonometric and hyperbolic functions. Interestingly, we see some important differences, e.g. the role of functions of e^x.

There are a number of issues and formulas left unresolved. For example, what are the results when $sinf(x)$ is expressed in terms of $\sin x$ and $\sinh x$? What is the meaning of $sinf(ix)$? In Part II of this work we will investigate formulas for the Fibonometric sine and cosine of the sum and difference of two real numbers. The method used will also provide insight into the Fundamental Identity.

Finally, the results of this investigation were obtained while the author was an undergraduate student at Huntingdon College. It was done under the direction of G. Joseph Wimbish who helped her chart the direction of the investigation. The results of this part of the work are those obtained in her Senior Capstone Project, while the results of Part II are those obtained in her Mathematics Honors Project.

Department of Mathematics
Clemson University
Clemson, SC 29632-0001
robynboo.bellsouth.com