A Trigonometric-Algebraic Analysis in a Problem Solving Situation

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Abstract. Teachers are always seeking interesting problem solving situations that employ a variety of mathematics content areas. We shall describe one such situation in which trigonometry and algebra are both central to the solution of the problem.

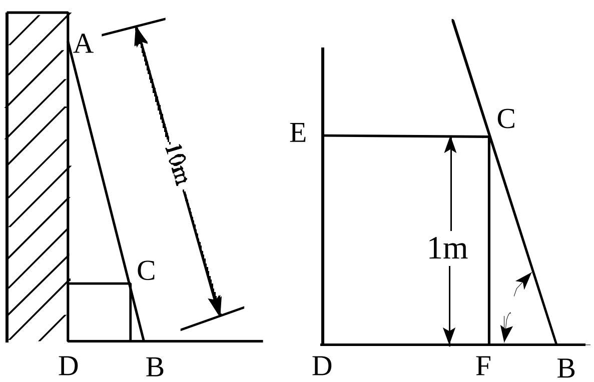

Figure 1

A 10-meter ladder, $\overline{AB}$, rests on the ground at point B and rests along the perpendicular wall at point A. It touches the corner, C, of a box that is one meter long and one meter high, and which sits in the corner of the building as shown in Figure 1. How high on the wall does the ladder rest; that is, what is the length of $\overline{AD}$?

To address this problem, we first find the measure of the acute angle made by the ground and the ladder, $\theta = m\angle FBC$. By corresponding angles of parallel lines, $\theta = m\angle ECA$. Furthermore, let $x = \overline{AC}$ and $y = \overline{CB}$.

Observe, in $\triangle CFB$,

$$\overline{CF} = y = \frac{1}{\sin(\theta)} \ .$$

Similarly, in $\triangle AEC$,

$$\overline{AE} = x = \frac{1}{\cos(\theta)} \ .$$

Then,

$$\overline{AB} = 10 = x + y = \frac{1}{\cos(\theta)} + \frac{1}{\sin(\theta)} = \frac{\sin(\theta) + \cos(\theta)}{\cos(\theta)\sin(\theta)} \ .$$

By squaring, we obtain

$$\begin{aligned} 100 &= \frac{\sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta)}{(\cos(\theta)\sin(\theta))^2} \\ &= \frac{1 + 2\sin(\theta)\cos(\theta)}{(\cos(\theta)\sin(\theta))^2} = \frac{1+\sin(2\theta)}{\left(\frac{1}{2}\sin(2\theta)\right)^2}, \end{aligned}$$

where we have made use of the double angle identity for the sine.

Therefore, we have

$$\frac{1+\sin(2\theta)}{\sin^2(2\theta)} = 25 \ .$$

If we substitute $a = \sin(2\theta)$, we are led to the quadratic equation $25a^2 - a - 1 = 0$, whose solutions are

$$a = \frac{1 \pm \sqrt{101}}{50} \ .$$

From the statement of the problem, θ is between $0°$ and $90°$ and thus 2θ is between $0°$ and $180°$, so $\sin(2\theta)$ is always positive for such angles. Hence, we can discard the negative solution and

$$a = \frac{1 \pm \sqrt{101}}{50} \approx 0.2210 = \sin(2\theta) \ .$$

Therefore, $2\theta \approx 12.8°$or $167.2°$, and so $\theta \approx 6.4°$ or $83.6°$.

We have obtained two acute angle solutions, even though Figure 1 only suggests one solution. These two solutions are compliments of each other and are $m\angle DBA$ and $m\angle DAB$. Although these solutions are distinct in the physical setting, they are indistinguishable if the physical context is removed from Figure 1.

We may now solve the original problem; that is, to find the length of $\overline{AD}$. In $\triangle ADB$, we get $\overline{AD} = 10\sin(6.4°) \approx 1.11$ meters, or $\overline{AD} = 10 \sin(83.6°) \approx 9.94$ meters. The reader may draw a picture of each solution.

Challenges:

(1) Find other methods of solving this problem.
(2) Find other situations which lend themselves to a trigonometric-algebraic analysis.

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