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Relating the Area and Perimeter of Right Triangles

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For a natural number n, we find the number of right triangles that have an area equal to n times perimeter. We begin by showing that the number of primitive right triangles for which twice the area is n times the perimeter is 2^k where k is the number of distinct odd primes in the canonical factorization of n. As a corollary we show that the number of primitive right triangles whose area is n times the perimeter is 2^k. We then use these results to provide a formula based on the canonical factorization of n for the number of right triangles whose area is n times the perimeter.

Introduction

A *Pythagorean triple* consists of three positive integers a, b, and c such that $a^2 + b^2 = c^2$. If the integers a, b, and c are relatively prime then the triple, (a, b, c), is called a *primitive Pythagorean triple*. The integers in a primitive Pythagorean triple make up the side lengths of what is called a *primitive right triangle*. Supriya Mohanty et. al. (1990) define a *primitive Pythagorean number* to be the area of a primitive right triangle. They show that all primitive Pythagorean numbers are divisible by 6; the product of three consecutive integers n, $(n + 1)$, and $(n + 2)$ with n odd is a primitive Pythagorean number; and that there are infinitely many primitive Pythagorean numbers which are products of two consecutive integers. Russell A. Gordon (2011) explores the properties of the semiperimeters (half the perimeters) of primitive right triangles. He uses techniques involving prime factorization to prove that for every positive integer k, there exist infinitely many positive integers s such that s is the semiperimeter of exactly k distinct primitive right triangles.

William Parsons (1984) introduces the idea of relating a triangle's area and perimeter, and N.J. Kuenzi and Bob Prielipp (1976) prove that for every natural number n there is at least one primitive right triangle whose area is n times its perimeter. A recent paper by P. Maynard (2005) explores the number of pythagorean triples whose area is n times the perimeter. Through the use of a technical lemma he derives this number and then as a corollary gives the number of primitive triples that satisfy the same condition. Here we take the opposite approach and begin with the number of primitive triples that satisfy a related condition involving twice the area. We then use the result on primitive triples to prove the more general case for non-primitive triangles, discovering a formula that agrees with the Maynard result.

Part of this work appeared in Ms. Short's and Mr. Yu Tiamco's senior thesis

We begin by reviewing a classical characterization of primitive Pythagorean triples. Next we prove our first theorem involving the number of primitive triples for which twice the area is n times the perimeter. As a corollary we obtain the number of primitive triples whose area is n times the perimeter in terms of the canonical factorization of n. In the last section we extend these results to non-primitive triples.

Euclid's Conditions

Let (a, b, c) be a primitive Pythagorean triple. It has been shown that exactly one of the legs of a primitive right triangle has an even length (Maynard, 2005); we let a be the even length and c be the length of the hypotenuse.

A standard number theory result states that for two relatively prime integers, α and β, with $\alpha < \beta$ and $\alpha + \beta$ odd, the triple given by

$$a = 2\alpha\beta, \qquad b = \beta^2 - \alpha^2, \qquad c = \beta^2 + \alpha^2 \tag{1}$$

is a primitive Pythagorean triple (Shockley, 1967). Furthermore, every primitive Pythagorean triple arises from a unique pair (α,β) (Shockley, 1967). Together these equations are also known as *Euclid's formulas*, and we will hereafter refer to the three conditions on α and β

- α and β are relatively prime,
- $\alpha < \beta$, and
- α and β have opposite parity

as *Euclid's conditions*.

Using Euclid's formulas, the perimeter and area of a primitive right triangle are given by:

$$A = \alpha\beta(\beta^2 - \alpha^2), \qquad P = 2\beta(\beta + \alpha) \tag{2}$$

for appropriate α and β.

On the Number of Primitive Triples with $A = nP$

We wish to count the number of primitive Pythagorean triples whose area is n times the perimeter. It will be convenient to work with the slightly altered condition $rA = nP$ for natural numbers r and n. We define $X(r,n)$ to be the number of primitive Pythagorean triples that satisfy $rA = nP$.

Using Euclid's conditions, we can transform the relation $rA = nP$ into a condition on α and β. We obtain

$$r\alpha(\beta - \alpha) = 2n. \tag{3}$$

We see that r must divide $2n$ in order for $X(r,n)$ to be nonzero. Setting $r = 2$ gives the most natural case which leads us to:

Definition 1. *Let n be a natural number. An* Area-Perimeter, or AP, pair *for n is a pair* (α,β) *such that*

- α *and* β *are relatively prime,*
- $\alpha < \beta$,
- α *and* β *have opposite parity, and*
- $\alpha(\beta - \alpha) = n$.

By Euclid's formulas, distinct AP pairs yield distinct primitive right triangles (Shockley, 1967). Furthermore each primitive right triangle gives rise to an AP pair. This one-to-one correspondence allows us to count the number of primitive Pythagorean triples that satisfy $2A = nP$ by counting the number of AP pairs.

We write $n = 2^m \prod_{i=1}^{k} p_i^{e_i}$ where p_i are distinct odd primes and the e_i are positive integers and k is the number of distinct odd primes in the canonical factorization of n. Substituting this value for n into (3) with $r = 2$ we obtain,

$$\alpha(\beta - \alpha) = 2^m \prod_{i=1}^{k} p_i^{e_i}. \tag{4}$$

By counting AP pair solutions to (4) we can find $X(2,n)$. When counting AP pairs, the following Lemmas become useful:

Lemma 1. *If* (α,β) *is an AP pair for n, the two terms* α *and* $(\beta - \alpha)$ *are relatively prime.*

Proof. Let $n \in \mathbb{N}$ and let (α,β) be an AP-pair for n. Assume α and $(\beta - \alpha)$ have a common divisor $q > 1$. Then $\alpha = qr$ and $\beta - \alpha = qs$ for some integers r and s. Solving for β, we obtain $\beta = qs + qr = q(s + r)$. Now α and β have a common divisor $q > 1$, a contradiction. □

Lemma 2. *If* (α,β) *is an AP pair for even n, then* α *is even.*

Proof. Let $n \in 2\mathbb{N}$ and let (α,β) be an AP-pair for n. Assume α is odd. By the definition of an AP-pair, the product $\alpha(\beta-\alpha) = n$ is even as n is even. But then $\beta-\alpha$ must be even which implies that α and β are not of opposite parity. □

We are now prepared for proving the main result of this section.

Theorem 1. *Let* $n \in \mathbb{N}$ *be given and write* $n = 2^m \prod_{i=1}^{k} p_i^{e_i}$ *where* p_i *are distinct odd primes. Then there are* 2^k *primitive triples which satisfy* $2A = nP$. *That is,*

$$X(2,n) = 2^k.$$

Proof. Let n be a natural number and write $n = 2^m \prod_{i=1}^{k} p_i^{e_i}$. We need to count the number of integral solutions to equation (4). By Lemma 1 if $p_i|\alpha$, then $p_i^{e_i}|\alpha$. Furthermore by Lemma 2 we know that $2^m|\alpha$. So our problem reduces to counting the number of ways of splitting the odd primes between α and $\beta - \alpha$. There are 2^k different ways of doing this. □

We also have the following result which is a direct corollary of Theorem 1.

Corollary 1. *For every natural number n where n has k distinct odd primes in its canonical factorization, there exist exactly* 2^k *distinct primitive right triangles whose area equals n times its perimeter.*

Proof. The condition $A = nP$ is equivalent to $2A = (2n)P$ and by Theorem 1 there are exactly 2^k primitive triangles which satisfy this condition. □

We note that appropriate integral solutions to (3) with $r = 1$ by way of (1) produce the primitive triples that satisfy $A = nP$. For example take $n = 7$. The relation $A = 7P$ is equivalent to $2A = 14P$ which implies we need AP pairs that solve $\alpha(\beta - \alpha) = 14$. There are two given by $(\alpha,\beta) = (2,9)$ and $(\alpha,\beta) = (14,15)$ which yield the primitive Pythagorean triples $(36,77,85)$ and $(420,29,421)$ respectively. Each of these triples' area is 7 times its perimeter.

Examining a list of Pythagorean triples, one may notice that the triples $(12,5,13)$ and $(8,6,12)$ have the property that their areas are equal to their respective perimeters; note that the latter triple is not primitive. This observation motivated William Parsons to look for other right triangles with this property. He proves a theorem that states that no other of these triangles exists (see (Parsons, 1984)) and invites the reader to prove the following:

For every natural number n, there is at least one primitive Pythagorean triangle in which the area equals n times the perimeter.

Kuenzi and Prielipp (1976) provided a proof by showing that the primitive Pythagorean triple $(8n^2 + 4n, 4n + 1, 8n^2 + 4n + 1)$ has an area equal to n times its perimeter for every natural number n.

Casting the Kuenzi and Prielipp result into our notation, we see that their triple corresponds to AP pair $(2n, 2n + 1)$. That is, an integral solution to (3) with $r = 1$ is given by $\alpha = 2n$ and $\beta - \alpha = 1$.

On the Number of Triples with $A = nP$

Maynard (2005) counts the number of Pythagorean triangles whose area is a multiple of the perimeter. He uses the Pythagorean Theorem directly to prove a technical lemma which gives relationships between the side lengths of the triangle and the multiplicative factor. He then uses these relations to count the number of Pythagorean triples whose area is n times the perimeter directly. As a direct consequence of this earlier work, Maynard then gives the number of primitive triples that satisfy $A = nP$.

We take an alternative approach towards the same results. We use our counts on the number of primitive triples that satisfy $2A = nP$ as a means of building towards the total number of triples (primitive or not) that satisfy $A = nP$.

Let natural number n be given. Suppose (a, b, c) is a Pythagorean Triple that satisfies $A = nP$ where A is the area and P is the perimeter. Then we can write

$$(a, b, c) = (ra', rb', rc')$$

with (a', b', c') primitive and r is the common scaling factor. Because (a, b, c) satisfies $A = nP$ we conclude that (a', b', c') satisfies

$$rA' = nP' \tag{5}$$

where A' and P' are the area and perimeter for the primitive triple. Furthermore by Euclid's conditions the primitive triple has an α' and β' that correspond to it. Plugging (2) into (5) and simplifying we obtain:

$$r\alpha'(\beta' - \alpha') = 2n.$$

We can conclude that if (a, b, c) satisfy $A = nP$ then the scaling factor r divides $2n$. Moreover, for this triple there exists a corresponding primitive triple which satisfies the related condition (5).

Alternatively, let r divide $2n$ and consider primitive triples that satisfy $2A = \frac{2n}{r}P$. Clearly these primitive triples can be scaled up to non-primitive triples for which the area is n times the perimeter. This leads us to:

Lemma 3. *A triple (a, b, c) satisfies $A = nP$ if and only if (a, b, c) satisfy the following conditions:*
1. $(a, b, c) = (ra', rb', rc')$ with (a', b', c') primitive.
2. $r|2n$.
3. $rA' = nP'$ where A' and P' are the area and perimeter of the associated primitive triple (a', b', c').

So to count the number of triples that satisfy $A = nP$ it suffices to count the number of primitive triples that satisfy the three conditions given in Lemma 3. As before, we define $X(r, n)$ to be the number of primitive Pythagorean triples that satisfy $rA = nP$ where A is the area and P is the perimeter of the corresponding right triangle.

Lemma 4. *Let r and n be integers so that $r|2n$ and let p be an odd prime that does not divide n. Let e be a positive integer and let i be an integer between 0 and e inclusive. Then*

$$X(p^i r, p^e n) = \begin{cases} X(r, n) & i = e, \\ 2X(r, n) & i < e. \end{cases}$$

Proof. The equation $p^i rA = p^e nP$ is equivalent to $2A = p^{e-i}\frac{2n}{r}P$. By Theorem 1, the number of primitive triangles that satisfy this last equation depends on the number of distinct odd primes in the canonical factorization of $p^{e-i}\frac{2n}{r}$. If $i = e$ then the number of distinct odd primes remains unchanged, if $i < e$ this number increases by one, meaning an additional power of 2. ◻

We are now ready for the main result.

Theorem 2. *Let $n = 2^m \prod_{i=1}^{k} p_i^{e_i}$. Then the number of Pythagorean triangles whose area is n times the perimeter is given by:*

$$(m + 2)\prod_{i=1}^{k}(2e_i + 1). \tag{6}$$

Proof. Let natural number n be given. We want to count the number of triples that satisfy $A = nP$. Using Lemma 3, for each factor r of $2n$ it suffices to count the number of primitives that satisfy $rA' = nP'$ where again A' and P' are the area and perimeter of the corresponding primitive. By Theorem 1, the number of these primitives depends on the number of distinct odd primes dividing $2n/r$. The difficulty comes in grouping the divisors of $2n$ in a meaningful way to make this calculation simple. To handle this difficulty we will use induction on k, the number of distinct odd primes in the canonical factorization of n.

For $k = 0$, we have to count the number of pythagorean triples which satisfy $A = 2^m P$. There are exactly $m + 2$ factors of $2n = 2^{m+1}$. For each of these factors, there is a unique primitive triple that satisfies $rA' = nP'$ by Theorem 1. This finishes the base case.

Now assume that for every n with $k-1$ distinct odd primes in its canonical factorization that equation (6) holds. Let n be a natural number with exactly k distinct odd primes in its canonical factorization. As before we write:

$$n = 2^m \prod_{i=1}^{k} p_i^{e_i}.$$

We isolate a single odd prime, say p_1 and define $u = n/p_1^{e_1}$ so that u has exactly $k-1$ distinct odd primes in its canonical factorization. Now the number of triples that satisfy $A = nP$ is given by:

$$\begin{aligned} \sum_{r|2n} X(r, n) &= \sum_{s|2u}\left(\sum_{i=0}^{e_1} X(sp_1^i, p_1^{e_1}u)\right) \\ &= \sum_{s|2u}(2e_1 + 1)X(s, u) \\ &= (2e_1 + 1)\sum_{s|2u} X(s, u). \end{aligned}$$

Here we have used Lemma 4. Now using the induction hypothesis we obtain the desired result. ◻

Maynard (2005) gives the same result in terms of $\tau(x)$, the multiplicative function that counts the divisors of x. Maynard proves that the number of Pythagorean triples that satisfy $A = nP$ is given by $\frac{1}{2}\tau(8n^2)$. Using the canonical factorization of n it is not difficult to show that our result agrees with the Maynard theorem.

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