Undergraduate Research

A Simple Delay Differential Equation

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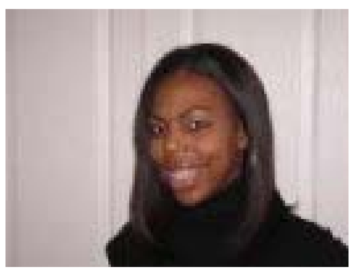
Biographical Sketch



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Kenya T. McLin, daughter of Eugene and Cassandra Jackson, is a native of Atlanta, Georgia and a 2005 graduate of Chamblee Charter High School in Chamblee, Georgia. She is currently a senior at Tuskegee University and will be receiving a Bachelor of Art degree in Mathematics on May 10, 2009. Kenya's post-graduation plans include working as an Operations Researcher for the United States Army Missile and Defense Command and attending graduate school at the University of Alabama at Huntsville pursuing

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a Master of Science degree in Mathematical Sciences. Her-long term goal is to be a mathematics professor at a Historically Black College or University. She is an active member of several clubs and organizations at Tuskegee University including being an Executive Board Member of *Alpha Kappa Alpha Sorority, Inc., Gamma Kappa Chapter* and the President of *Kappa Delta Pi International Honor Society in Education, Lambda Delta Chapter.* Kenya also received the prestigious honor of being the University Scholar for the College of Liberal Arts and Education. This paper was written under the direction and supervision of Dr. Hussain Elalaoui-Talibi, her summer research advisor at Tuskegee University.

Abstract: We give a simple solution of a classical linear delay differential equation.

Most students who have taken an undergraduate course in differential equations have never heard of a delay differential equation. This should not be the case, especially since the problem of mixing salt brines encountered in undergraduate differential equations texts, is modeled by a delay differential equation if one does not assume instantaneous perfect mixing. Our goal in this note is to give the general solution of a simple linear differential equation, using calculus and mathematical induction. The solution we give is efficient from a computational point of view, and uses the known method of steps.

The next lemma is a simple result about iterated integrals that will make the computation of the solutions easier. It can be easily proved using calculus and mathematical induction, by changing the order of integration.

LEMMA 1. *If $g(x)$ is continuous on the interval $[a, b]$ and we define $I_0(x) = \int_a^x g(t)dt$ and*

$$I_{n+1}(x) = \int_a^x I_n(t)dt \quad \text{for } n = 0, 1, 2, ...,$$

then

$$I_n(x) = \frac{1}{n!}\int_a^x (x-t)^n g(t)dt \quad \text{for } n = 0, 1, 2, ... \quad .$$

We next state a classical simple first order linear delay differential equation and derive a general solution for it. Since the solution will be given piecewise on the intervals $[0, 1]$, $[1, 2]$, ..., the previous lemma will be useful when one uses this for different initial conditions.

THEOREM 1. *The solution of the equation*

$$y'(t) = ay(t) + by(t-1), t \geq 0$$

$$y(t) = f(t), t \in [-1, 0],$$

where f is continuous on $[-1, 0]$, is

$$y(t) = p_k(t)e^{a(t-k)} + b^{k+1}e^{a(t-k-1)}I_k(t-k-1),$$

where $k = 0, 1, 2, 3, ...$ and $k \leq t \leq k+1$, and

$$p_k(t) = \sum_{j=0}^{k} \frac{y(j)b^{k-j}}{(k-j)!}(t-k)^{k-j}$$

$$I_0(t) = \int_{-1}^{t} f(s)e^{-as}ds$$

$$I_{n+1}(t) = \int_{-1}^{t} I_n(s)ds.$$

Proof: The equation will be solved piecewise on the intervals $k \leq t \leq k+1$, where $k = 0, 1, 2, 3, ...$ We will use a mathematical induction argument to establish the solution.

Solution on $[0, 1]$: Here, the problem reduces to

$$y'(t) = ay(t) + bf(t-1).$$

We have

$$(y(t)e^{-at})' = bf(t-1)e^{-at}$$

$$\int_0^t (y(s)e^{-as})'ds = b\int_0^t f(s-1)e^{-as}ds$$

$$\begin{aligned}[y(s)e^{-as}]_0^t &= b\int_{-1}^{t-1} f(u)e^{-a(u+1)}du \\ &= be^{-a}\int_{-1}^{t-1} f(u)e^{-au}du,\end{aligned}$$

$$y(t)e^{-at} - y(0) = be^{-a}I_0(t-1)$$

$$\begin{aligned}y(t) &= y(0)e^{at} + be^{a(t-1)}I_0(t-1) \\ &= p_0e^{at} + be^{a(t-1)}I_0(t-1).\end{aligned}$$

General Form of the Solution

We will use an induction argument. Assume that for all $n \leq k$,

$$y(t) = p_n(t)e^{a(t-n)} + b^{n+1}e^{a(t-n-1)}I_n(t-n-1),$$

where $n \leq t \leq n+1$. The problem on the interval $k+1 \leq t \leq k+2$ is then

$$y'(t) = ay(t) + bp_k(t-1)e^{a(t-k-1)} + b^{k+1}e^{a(t-k-2)}I_k(t-k-2).$$

Now,

$$(e^{-at}y)' = bp_k(t-1)e^{-a(k+1)} + b^{k+1}e^{-a(k+2)}I_k(t-k-2).$$

$$\begin{aligned}y(t)e^{-at} - y(k+1)e^{-a(k+1)} &= be^{-a(k+1)}\textstyle\int_{k+1}^t p_k(s-1)ds \\ &+ b^{k+1}e^{-a(k+2)}\textstyle\int_{k+1}^t I_k(s-k-2)ds.\end{aligned}$$

Considering the last two integrals on the right hand side of the last equation separately,

$$\begin{aligned}\int_{k+1}^t p_k(s-1)ds &= \sum_{j=0}^k \frac{y(j)b^{k-j}}{(k-j)!}\int_{k+1}^t (s-k-1)^{k-j}ds \\ &= \sum_{j=0}^k \frac{y(j)b^{k-j}}{(k-j+1)!}(t-k-1)^{k-j+1}.\end{aligned}$$

In the next integral, we use the substitution $u = s - k - 2$, to get

$$\begin{aligned}\int_{k+1}^{t} I_k(s-k-2)ds &= \int_{-1}^{t-k-2} I_k(u)du \\ &= I_{k+1}(t-k-2).\end{aligned}$$

Hence,

$$\begin{aligned} y(t)e^{-at} &- y(k+1)e^{-a(k+1)} \\ &= be^{-a(k+1)} \textstyle\sum_{j=0}^{k} \frac{y(j)b^{k-j}}{(k-j+1)!}(t-k-1)^{k-j+1} \\ &+ b^{k+1}e^{-a(k+2)} I_{k+1}(t-k-2) \\ y(t) &= e^{a[t-(k+1)]}[y(k+1) \\ &+ \textstyle\sum_{j=0}^{k+1} \frac{y(j)b^{(k+1)-j}}{[(k+1)-j]!}\left[t-(k+1)\right]^{(k+1)-j}] \\ &+ b^{k+1}e^{a[t-(k+2)]} I_{k+1}(t-k-2)\end{aligned}$$

$y(t) = p_{k+1}(t)e^{a[t-(k+1)]} + b^{k+1}e^{a[(t-(k+1)-1]} I_{k+1}(t-(k+1)-1)$, which establishes the induction step, and finishes the proof.

Example 1 By letting $a = 0$, $b = -1$, and $f(t) = 1$, one gets the equation

$$y'(t) = -y(t-1), t \geq 0$$

$$y(t) = 1, t \in [-1, 0],$$

Applying our results, it is easily seen that the solution in the interval $[0, 1]$ is

$$y(t) = 1 - t$$

while the solution in the interval $[1, 2]$ is

$$y(t) = -2t + \frac{1}{2}t^2 + \frac{3}{2}.$$

Example 2

A more complicated example that requires integration by parts is the following exercise, which is problem 3, from [1] p. 213:

$$y'(t) = ay(t) + by(t-1), t \geq 0$$

$$y(t) = 1 + t, t \in [-1, 0],$$

Applying our results, it is easily seen that the solution in the interval $[0, 1]$ is

$$y(t) = \frac{-b}{a^2} - \frac{bt}{a} + \left(1 + \frac{b}{a^2}\right) e^{at}$$

while the solution in the interval $[1, 2]$ is

$$\begin{aligned} y(t) &= \tfrac{2b^2}{a^3} - \tfrac{b^2}{a^2} + \tfrac{b^2 t}{a^2} \\ &+ \left[1 + \tfrac{b}{a^2} - (b + \tfrac{b}{a} + \tfrac{b}{a^2} + \tfrac{b^2}{a^2})e^{-a}\right] e^{at} \\ &- \tfrac{2b^2}{a^3} e^{a(t-1)} + \left(b + \tfrac{b^2}{a^2}\right) t e^{a(t-1)}. \end{aligned}$$

References

[1] R. D. Driver, *Introduction to Ordinary Differential Equations,* Harper & Row, New York, 1978.

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