# Undergraduate Research

## On the Point of Intersection of Two Lines in Space

By Erica Thompson and Hussain Elalaoui-Talibi

### Biographical Sketch

Erica Thompson

Erica Dawn Thompson is the daughter of Mrs. Sandra K. Moore of Detroit, MI, and a 2004 graduate of Lewis Cass Technical High School in Detroit, Michigan. Currently a senior at Tuskegee



Research for this paper was completed at the NREUP Summer 2008 Research Experience for Undergraduates at Tuskegee University, under the auspices of MAA grant no. 39-22153006

University, she will be receiving a B.S. in Electrical Engineering in May 2010. Erica's immediate plans include becoming a Peace Corps volunteer, so that she can give to those who can benefit from her knowledge, and so that she can also benefit from theirs. Erica is a very active student at Tuskegee University, and has held many positions. Erica is currently the 2008-2009 Chair of the *Engineering Student Leadership Council* in Tuskegee's College of Engineering, Architecture, and Physical Sciences, and also served as president of *Alpha Kappa Mu National Honor Society* on Tuskegee's campus for the 2007-2008 school year. This paper was written under the direction and supervision of Dr. Hussain Talibi, her program advisor at Tuskegee University.

**Abstract**: We give an explicit formula for the point of intersection of two lines in space.

A common exercise in second or third semester Calculus books is to determine if two lines in space are parallel, skew, or intersecting; and if they intersect, to find the point of intersection. The students usually have to solve a system of 3 equations in 2 variables to answer these questions. Even though that is simple to do, it is inefficient from a computational point of view. Our purpose in this note is to give a criterion that tells if two lines intersect, and if they do, we give an explicit formula for the point of intersection.

A line $L$ in $\mathbb{R}^3$ is determined by a point $P_0 = (x_0, y_0, z_0)$ on the line and a vector $\mathbf{v}$ parallel to the line. We will think of $P_0$ as a vector $\mathbf{x}_0$. The length of a vector $\mathbf{v}$ will be denoted by $|\mathbf{v}|$, while the dot and cross products of two vectors $\mathbf{v}$, and $\mathbf{v}'$ will be denoted respectively by, $\mathbf{v} \cdot \mathbf{v}'$, and $\mathbf{v} \times \mathbf{v}'$. The next lemma is a simple result that gives a condition which guarantees that two lines are skew.

Lemma 1. *The lines $L$: $\mathbf{x}_0, \mathbf{v}$ and $L'$: $\mathbf{x}'_0, \mathbf{v}'$ in $\mathbb{R}^3$ lie in the same plane if and only if*

$$(\mathbf{x}_0 - \mathbf{x}'_0) \cdot (\mathbf{v} \times \mathbf{v}') = 0.$$

*In particular, the lines $L$: $\mathbf{x}_0, \mathbf{v}$ and $L'$: $\mathbf{x}'_0, \mathbf{v}'$ are skew if and only if $(\mathbf{x}_0 - \mathbf{x}'_0) \cdot (\mathbf{v} \times \mathbf{v}') \neq 0$.*

*Proof*: First assume that $L$ and $L'$ lie in the same plane. If $L$ and $L'$ are parallel, then the equation is trivial since $\mathbf{v} \times \mathbf{v}' = \mathbf{0}$, so assume that $L$ and $L'$ are not parallel. It follows that $(\mathbf{v} \times \mathbf{v}')$ is perpendicular to the plane containing $L$ and $L'$. Since $\mathbf{x}_0 - \mathbf{x}'_0$ lies in the plane containing $L$ and $L'$, it follows that $(\mathbf{x}_0 - \mathbf{x}'_0) \cdot (\mathbf{v} \times \mathbf{v}') = 0$.

Now assume that $(\mathbf{x}_0 - \mathbf{x}'_0) \cdot (\mathbf{v} \times \mathbf{v}') = 0$, and consider the plane with equation $(\mathbf{x}_0 - \mathbf{x}'_0) \cdot (\mathbf{v} \times \mathbf{v}') = 0$. This plane clearly contains $L'$. We will show that it also contains $L$. Writing $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$ for the equation of $L$, we have:

$$
\begin{aligned}
(\mathbf{x}(t)-\mathbf{x}_0')\cdot(\mathbf{v}\times\mathbf{v}') &= (\mathbf{x}_0-\mathbf{x}_0'+t\mathbf{v})\cdot(\mathbf{v}\times\mathbf{v}')\\
&= (\mathbf{x}-\mathbf{x}_0')\cdot(\mathbf{v}\times\mathbf{v}')+t\mathbf{v}\cdot(\mathbf{v}\times\mathbf{v}')\\
&= 0+0=0,
\end{aligned}
$$

which finishes the proof.

The following corollary gives a condition that guarantees that two lines intersect.

COROLLARY 1. *If the lines* $L$*:* $\mathbf{x}_0,\mathbf{v}$ *and* $L'$*:* $\mathbf{x}_0',\mathbf{v}'$ *in* $\mathbb{R}^3$ *are such that* $\mathbf{v}\times\mathbf{v}'\neq\mathbf{0}$*, and* $(\mathbf{x}_0-\mathbf{x}_0')\cdot(\mathbf{v}\times\mathbf{v}')=0$*, then* $L\cap L'\neq\emptyset$

*Proof*: From the previous lemma, we know that the lines lie in the same plane, so they are either parallel or they intersect. Since $\mathbf{v}\times\mathbf{v}'\neq\mathbf{0}$, the lines can't be parallel, so they must intersect, and that completes the proof.

The next result gives an explicit formula for the point of intersection of two lines that satisfy the conditions of the last corollary. For this result, we let

$$\hat{\mathbf{v}}=\frac{\mathbf{v}}{|\mathbf{v}|},$$

and

$$\hat{\mathbf{v}}_\perp=\frac{\mathbf{v}'-(\mathbf{v}'\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}}{|\mathbf{v}'-(\mathbf{v}'\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}|}.$$

Note that $\hat{\mathbf{v}}$ is the unit vector in the direction of $\mathbf{v}$, while $\hat{\mathbf{v}}_\perp$ is the component of $\mathbf{v}'$ orthogonal to $\mathbf{v}$.

THEOREM 1. *If the lines* $L$*:* $\mathbf{x}_0,\mathbf{v}$ *and* $L'$*:* $\mathbf{x}_0',\mathbf{v}'$ *in* $\mathbb{R}^3$ *are such that* $L\cap L'\neq\emptyset$*, then the point of intersection is given by*

$$\mathbf{x}_0+\frac{[(\mathbf{x}_0'-\mathbf{x}_0)\cdot\hat{\mathbf{v}}](\mathbf{v}'\cdot\hat{\mathbf{v}}_\perp)-[(\mathbf{x}_0'-\mathbf{x}_0)\cdot\hat{\mathbf{v}}_\perp](\mathbf{v}'\cdot\hat{\mathbf{v}})}{\mathbf{v}'\cdot\hat{\mathbf{v}}_\perp}\hat{\mathbf{v}}$$

*Proof*: If $\mathbf{x}_0=\mathbf{x}_0'$, the result is trivial, so assume that $\mathbf{x}_0\neq\mathbf{x}_0'$. First, we note that since $(\mathbf{x}_0-\mathbf{x}_0')\cdot(\mathbf{v}\times\mathbf{v}')=0$, the vector $\mathbf{x}_0-\mathbf{x}_0'$ lies in the plane spanned by $\mathbf{v}$ and $\mathbf{v}'$. Writing $\mathbf{v}'=v_1'\hat{\mathbf{v}}+v_2'\hat{\mathbf{v}}_\perp$, we may write $\mathbf{x}_0'-\mathbf{x}_0=a\hat{\mathbf{v}}+b\hat{\mathbf{v}}_\perp$. Translating the origin to $\mathbf{x}_0$, we may write the equations of the lines in the new system as

$$\mathbf{x}(s)=s|\mathbf{v}|\hat{\mathbf{v}}$$

$$\mathbf{x}(t)=\mathbf{x}_0'-\mathbf{x}_0+t\mathbf{v}'=(a+v_1't)\hat{\mathbf{v}}+(b+v_2't)\hat{\mathbf{v}}_\perp.$$

The point of intersection of the lines is then given by

$$s|\mathbf{v}|=a+v_1't$$

$$0=b+v_2't.$$

The solutions are

$$s = \frac{av_2' - bv_1'}{v_2'|\mathbf{v}|}$$
$$t = -\frac{b}{v_2'}$$

These solutions are clearly consistent, and the point of intersection is then

$$\mathbf{x}_0 + \mathbf{x}(s) = \mathbf{x}_0 + \frac{av_2' - bv_1'}{v_2'}\hat{\mathbf{v}},$$

where the various parameters are given by

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$
$$\hat{\mathbf{v}}_\perp = \frac{\mathbf{v}' - (\mathbf{v}' \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}}{|\mathbf{v}' - (\mathbf{v}' \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}|}$$
$$v_1' = \mathbf{v}' \cdot \hat{\mathbf{v}}$$
$$v_2' = \mathbf{v}' \cdot \hat{\mathbf{v}}_\perp$$
$$a = (\mathbf{x}_0' - \mathbf{x}_0) \cdot \hat{\mathbf{v}}$$
$$b = (\mathbf{x}_0' - \mathbf{x}_0) \cdot \hat{\mathbf{v}}_\perp.$$

Using these, the point of intersection is then given by

$$\mathbf{x}_0 + \frac{[(\mathbf{x}_0' - \mathbf{x}_0) \cdot \hat{\mathbf{v}}](\mathbf{v}' \cdot \hat{\mathbf{v}}_\perp) - [(\mathbf{x}_0' - \mathbf{x}_0) \cdot \hat{\mathbf{v}}_\perp](\mathbf{v}' \cdot \hat{\mathbf{v}})}{\mathbf{v}' \cdot \hat{\mathbf{v}}_\perp}\hat{\mathbf{v}},$$

which finishes the proof.

## References

[1] Stewart, J., 2008, *Calculus,* Thomson, Brooks/Cole.

Department of Mathematics
Tuskegee University
Tuskegee, AL 36088
talibi@tuskegee.edu
(334)-727-8212