

On the Form of Odd Perfect Gaussian Integers

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Introduction

We define $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ to be the set of *Gaussian integers* and we define *rational integers* to be elements of \mathbb{Z} . The set $\mathbb{Z}[i]$ is a ring and, in fact, a Euclidean domain under the usual operations of addition and multiplication. All Gaussian integers will be represented by Greek letters and rational integers by ordinary Latin letters. Primes will be denoted by π and p respectively. Units will be denoted by $\varepsilon = \pm 1, \pm i$ and 1 respectively. In 1961 Robert Spira [3] defined the sum-of-divisors function on $\mathbb{Z}[i]$ as follows. Let $\eta = \varepsilon \prod \pi_i^{k_i}$ be a Gaussian integer. This representation is unique in that we will choose our unit ε such that each π_i is in the first quadrant ($\operatorname{Re}(\pi_i) > 0$ and $\operatorname{Im}(\pi_i) \geq 0$). Spira defined the sum-of-divisors function σ as

$$\sigma(\eta) = \prod \frac{\pi_i^{k_i+1} - 1}{\pi_i - 1}$$

This can be seen to match precisely with the rational form of the sum-of-divisors exactly when the rational primes coincide with the Gaussian primes (when $p \equiv 3 \pmod{4}$).

A Gaussian integer η is considered *even* if and only if $1 + i$ divides η . It is easy to see that $1 + i$ divides $a + bi$ if and only if a and b have the same parity, i.e., $a \equiv b \pmod{2}$. It follows that the usual parity rules for addition for \mathbb{Z} hold in $\mathbb{Z}[i]$: The sum of two Gaussian integers of the same parity is even, and the sum of two Gaussian integers of opposite parity is odd. Also, since $a \equiv a^2 \pmod{2}$ for any rational integer a , it follows that a Gaussian integer

[25]

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$a + bi$ is even if and only if its norm, $N(a + bi) = a^2 + b^2$, is an even rational integer.

Finally, we can define *perfect numbers* in the natural way. A Gaussian integer η is *perfect* if $\sigma(\eta) = (1 + i)\eta$. This is not the only notion we can work with, though. The Gaussian integer η is *norm-perfect* if $N(\sigma(\eta)) = N(1 + i)N(\eta) = 2N(\eta)$. Every perfect number is norm-perfect, so often it is easier to work with the norm-perfect concept. Wayne McDaniel [1] proved a theorem for Gaussian integers analogous to Euclid's and Euler's characterization of even perfect positive integers. We will prove a theorem for Gaussian integers analogous to Euler's partial characterization of odd perfect positive integers, which is that any such integer must be of the form $p^j m^2$ where $\gcd(p, m) = 1$ and $p \equiv j \equiv 1 \pmod{4}$.

The Form of Odd Perfect Numbers

LEMMA 1. *If π is an odd prime, then $\sigma(\pi^m)$ is even if and only if m is odd.*

PROOF. We first claim that for any Gaussian integer η_j we have $N\left(\sum_j \eta_j\right) \equiv \sum_j N(\eta_j) \pmod{2}$. This follows from basic parity rules. We have $N\left(\sum_j \eta_j\right) \equiv 0 \pmod{2}$ if and only if $\sum_j \eta_j$ is even in the Gaussian sense, but this happens if and only if there are an even number of odd η_j 's. This means that there are an even number of $N(\eta_j) \equiv 1 \pmod{2}$. Therefore, we have that $N\left(\sum_j \eta_j\right) \equiv 0 \pmod{2}$ if and only if $\sum_j N(\eta_j) \equiv 0 \pmod{2}$.

Suppose π is an odd prime. Now examine $N(\sigma(\pi^m)) = N(1 + \pi + \cdots + \pi^m) \equiv N(1) + N(\pi) + \cdots + N(\pi^m) \pmod{2}$. We clearly have that this sum is congruent to $0 \pmod{2}$ if and only if m is odd since each term in the sum is odd. \square

THEOREM 1. *If α is an odd norm-perfect Gaussian integer, then $\alpha = \pi^k \gamma^2$, where k is an odd rational integer and $\gcd(\pi, \gamma) = \varepsilon$.*

PROOF. Let α be an odd norm-perfect Gaussian integer. Then $\alpha = \prod_{i=1}^n \pi_i^{k_i}$, where no π_j is associate to $(1+i)$. Since α is norm-perfect we have $N(\sigma(\alpha)) = 2N(\alpha)$, but since $N(\alpha)$ is odd and the sum of two squares we have that $N(\alpha) \equiv 1 \pmod{4}$, so $N(\sigma(\alpha)) \equiv 2N(\alpha) \equiv 2 \pmod{4}$.

Examine

$$\begin{aligned} N(\sigma(\alpha)) &= \prod_{i=1}^n N\left(\sigma\left(\pi_i^{k_i}\right)\right) \\ &= N\left(\sigma\left(\pi_1^{k_1}\right)\right) \cdots N\left(\sigma\left(\pi_n^{k_n}\right)\right) \end{aligned}$$

Without loss of generality we can suppose that $N\left(\sigma\left(\pi_1^{k_1}\right)\right) \equiv 2 \pmod{4}$ and all the other terms in the product above are congruent to $1 \pmod{4}$. This is because no term can be $0 \pmod{4}$ or else the whole product would be $0 \pmod{4}$, and no term can be $3 \pmod{4}$ since the norm is the sum of two squares. From Lemma 1 we know that k_1 is odd and that each k_i is even for all $1 < i \leq n$.

The form of an odd norm-perfect Gaussian integer then must have one prime to an odd power and the rest of the factorization are squares. Thus the form is $\alpha = \pi^k \gamma^2$ where k is odd and $\gcd(\pi, \gamma) = 1$. \square

COROLLARY 1. *If α is an odd perfect Gaussian integer, then $\alpha = \pi^k \gamma^2$, where k is an odd rational integer and $\gcd(\pi, \gamma) = \varepsilon$.*

PROOF. Every perfect Gaussian integer is norm-perfect. \square

1. Some Pesky Counterexamples

As noted above, in the rational case, Euler proved that every odd perfect number has the form $p^j b^2$ where p is a prime, $\gcd(p, b) = 1$, $p \equiv 1 \pmod{4}$, and $j \equiv 1 \pmod{4}$. In our form, $\pi_1^k \gamma^2$, we get that $N(\pi_1) \equiv 1 \pmod{4}$ for free since the norm is a sum of two squares. However, we cannot show that $k \equiv 1 \pmod{4}$.

It is still unknown whether or not there are any odd perfect Gaussian integers, but there are odd norm-perfect numbers. In fact the smallest is a rather disturbing example: $2+i$. It turns out that this is not only odd, but a prime. However, we shall show that this is the only such example. Let π be a Gaussian prime. If π is norm-perfect, then $N(\pi+1) = 2N(\pi)$; if $\pi = a+bi$ then this equation becomes

$$(a + 1)^2 + b^2 = 2(a^2 + b^2),$$

which is equivalent to $(a - 1)^2 + b^2 = 2$. There are only 4 integer solutions to this equation: $(2, -1)$, $(0, -1)$, $(2, 1)$, $(0, 1)$. Therefore, in the Gaussian integers, $2 + i$ and $2 - i$ are the only primes that are norm-perfect.

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References

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