On the Solution to Nonic Equations

By Dr. Raghavendra G. Kulkarni

Abstract. This paper presents a novel decomposition technique in which a given nonic equation is decomposed into quartic and quintic polynomials as factors, eventually leading to its solution in radicals. The conditions to be satisfied by the roots and the coefficients of such a solvable nonic equation are derived.

Introduction

It is well known that the general polynomial equations of degree five and above cannot be solved in radicals, and have to be solved algebraically using symbolic coefficients, such as the use of Bring radicals for solving the general quintic equation, and the use of Kampe de Feriet functions to solve the general sextic equation [1, 2]. There is hardly any literature on the analytical solution to the polynomial equations beyond sixth degree. The solution to certain solvable octic equations is described in a recent paper [4].

This paper presents a method of determining all nine roots of a nonic equation (ninth-degree polynomial equation) in radicals. First, the given nonic equation is decomposed into two factors in a novel fashion. One of these factors is a fourth-degree polynomial, while the other one is a fifth-degree polynomial. When the fourth-degree polynomial factor is equated to zero, the four roots of the given nonic equation are extracted by solving the resultant quartic equation using the well-known Ferrari method or by a method given in a recent paper [3]. However, when the fifth-degree polynomial factor is equated to zero, the resultant quintic equation is also required to be solved in radicals, since the aim of this paper is to solve the nonic equation completely in radicals. For this purpose, suitable supplementary equations are employed so that the given nonic equation can be successfully decomposed, and then the resultant quintic factor also can be further decomposed. The criteria to be satisfied by the roots and the coefficients of such a solvable nonic equation are derived. At the end, we solve a numerical example using the proposed method.

Decomposition of a Nonic Equation

Consider the following nonic equation in x :

$$x^9+a_8x^8+a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0, \tag{1}$$

where $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$, and a_8 are real coefficients. In the proposed method, we add a root "at the origin" to the above equation, which is equivalent to multiplying (1) by x. This action converts the nonic equation (1) to a decic equation as shown below:

$$x^{10}+a_8x^9+a_7x^8+a_6x^7+a_5x^6+a_4x^5+a_3x^4+a_2x^3+a_1x^2+a_0x=0. \tag{2}$$

Our aim is to decompose the above decic equation into two quintic factors. For this purpose, consider another decic equation as shown below:

$$\begin{aligned}\big(x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\big)^2 \\ -\left(c_4x^4+c_3x^3+c_2x^2+c_1x+c_0\right)^2 \quad &= 0,\end{aligned} \tag{3}$$

where b_0, b_1, b_2, b_3, b_4, and c_0, c_1, c_2, c_3, c_4 are the unknown coefficients in the quintic and the quartic polynomial terms respectively, in (3). Notice that the above decic equation (3) can be easily decomposed into two quintic factors as shown below:

$$\begin{aligned}&\big[\big(x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\big) \\ &\quad -\big(c_4x^4+c_3x^3+c_2x^2+c_1x+c_0\big)\big] \\ &*\big[\big(x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\big) \\ &\quad +\big(c_4x^4+c_3x^3+c_2x^2+c_1x+c_0\big)\big] \quad = 0,\end{aligned} \tag{4}$$

where the symbol * is the multiplication symbol. Equation 4 is further simplified as:

$$\begin{aligned}&\left[x^5+(b_4-c_4)x^4+(b_3-c_3)x^3+(b_2-c_2)x^2\right.\\&\qquad\qquad\left.+(b_1-c_1)x+b_0-c_0\right]\\&*\left[x^5+(b_4+c_4)x^4+(b_3+c_3)x^3+(b_2+c_2)x^2\right.\\&\qquad\qquad\left.+(b_1+c_1)x+b_0+c_0\right]\quad=0.\end{aligned} \tag{5}$$

Thus if the decic equation (2) can be represented in the form of decic equation (3), then it can be easily decomposed into two quintic factors as given in (5). In order for (3) to represent decic equation (2), the coefficients of (2) and (3) must be equal. Note, however, that the coefficients of (3) are not given explicitly. Therefore, to facilitate comparison of coefficients of (2) and (3), the decic equation (3) is expanded and rearranged in descending powers of x, as shown below:

$$\begin{aligned}&x^{10}+2b_4x^9+\left(b_4^2+2b_3-c_4^2\right)x^8+2\left(b_2+b_3b_4-c_3c_4\right)x^7\\&+\left(b_3^2+2b_1+2b_2b_4-c_3^2-2c_2c_4\right)x^6\\&+2\left(b_0+b_1b_4+b_2b_3-c_1c_4-c_2c_3\right)x^5\\&+\left(b_2^2+2b_0b_4+2b_1b_3-c_2^2-2c_0c_4-2c_1c_3\right)x^4\\&+2\left(b_0b_3+b_1b_2-c_0c_3-c_1c_2\right)x^3\\&+(b_1^2+2b_0b_2-c_1^2-2c_0c_2)x^2+2(b_0b_1-c_0c_1)x+b_0^2-c_0^2\quad=0.\end{aligned} \tag{6}$$

Equating the coefficients of (2) and (6), we obtain the following ten equations:

$$2b_4=a_8 \tag{7}$$

$$b_4^2+2b_3-c_4^2=a_7 \tag{8}$$

$$2\,(b_2 + b_3 b_4 - c_3 c_4) = a_6 \tag{9}$$

$$b_3^2 + 2b_1 + 2b_2 b_4 - c_3^2 - 2c_2 c_4 = a_5 \tag{10}$$

$$2\,(b_0 + b_1 b_4 + b_2 b_3 - c_1 c_4 - c_2 c_3) = a_4 \tag{11}$$

$$b_2^2 + 2b_0 b_4 + 2b_1 b_3 - c_2^2 - 2c_0 c_4 - 2c_1 c_3 = a_3 \tag{12}$$

$$2\,(b_0 b_3 + b_1 b_2 - c_0 c_3 - c_1 c_2) = a_2 \tag{13}$$

$$b_1^2 + 2b_0 b_2 - c_1^2 - 2c_0 c_2 = a_1 \tag{14}$$

$$2\,(b_0 b_1 - c_0 c_1) = a_0 \tag{15}$$

$$b_0^2 - c_0^2 = 0. \tag{16}$$

Equation 16 results into two expressions for c_0 as: $c_0 = \pm b_0$, and we choose:

$$c_0 = b_0. \tag{17}$$

When (17) is used in the factored decic equation (5), we obtain the following expression containing the quartic polynomial and the quintic polynomial as factors:

$$\begin{aligned} x\,\big[x^4 + (b_4 - c_4)\,x^3 + (b_3 - c_3)\,x^2 \\ + (b_2 - c_2)\,x + (b_1 - c_1)\big] \\ *\,\big[x^5 + (b_4 + c_4)\,x^4 + (b_3 + c_3)\,x^3 \\ + (b_2 + c_2)\,x^2 + (b_1 + c_1)\,x + 2b_0\big] \quad = 0. \end{aligned}$$

(18)

Notice that the added root ("at the origin," $x = 0$) to the given nonic equation (1) is visible in the factored equation (18). To facilitate extraction of the nine roots of the given nonic (1), the quartic and the quintic factors in (18) are equated to zero leading to the following quartic and quintic equations respectively as shown below:

$$x^4 + (b_4 - c_4)x^3 + (b_3 - c_3)x^2 + (b_2 - c_2)x + b_1 - c_1 = 0 \quad (19)$$

$$x^5 + (b_4 + c_4)x^4 + (b_3 + c_3)x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + 2b_0 = 0. \quad (20)$$

Once the unknown coefficients in (19) are determined, the quartic equation (19) can be solved in radicals by well-known methods [2, 3]. However, the same cannot be said for the quintic equation (20), since there is no solution in radicals for a general quintic equation. Thus to make quintic equation (20) solvable in radicals, we need to find some way to decompose it into factors. In the next section we introduce some supplementary equations to facilitate determination of unknowns as well as the decomposition of (20).

Choice of Supplementary Equations

Since general polynomial equations beyond fourth-degree cannot be solved in radicals, it is to be noted that, even though there are ten equations, (7) – (16), to determine the ten unknowns in the decic equation (3), some more equations (called supplementary equations) are required to determine all the ten unknowns and to decompose the quintic equation (20). The supplementary equations introduced will decide the type of solvable nonic equation. Let the supplementary equations introduced here be as indicated below:

$$c_4 = -b_4 \quad (21)$$

$$c_1 = -b_1 \quad (22)$$

Using these supplementary equations in the quartic and the quintic equations, (19) and (20), leads to the following equations:

$$x^4 + 2b_4x^3 + (b_3 - c_3)x^2 + (b_2 - c_2)x + 2b_1 = 0 \quad (23)$$

$$x^5 + (b_3 + c_3)x^3 + (b_2 + c_2)x^2 + 2b_0 = 0. \quad (24)$$

We decompose the quintic equation (24) as shown below:

$$x^3(x^2+b_3+c_3)+(b_2+c_2)\left(x_2+\frac{2b_0}{b_2+c_2}\right)=0. \qquad (25)$$

The above equation can be decomposed if the following condition is satisfied.

$$b_3+c_3=\frac{2b_0}{b_2+c_2} \qquad (26)$$

With the coefficients of the quintic equation (24) satisfying the condition (26), the equation (24) can be factored as shown below:

$$(x^2+b_3+c_3)(x^3+b_2+c_2)=0. \qquad (27)$$

The roots of the factored quintic equation (27) can be easily determined in radicals. The next task is to determine the unknowns $(b_0, b_1, b_2, b_3, b_4, c_0, c_1, c_2, c_3, c_4)$ using the equations, (7) – (16), and the supplementary equations (21), (22), and (26).

Determination of Unknowns

Using equations, (7), (8), and the supplementary equation (21), the following unknowns are determined as shown below:

$$b_4=\frac{a_8}{2}$$

$$c_4=-\frac{a_8}{2}$$

$$b_3=\frac{a_7}{2}. \qquad (28)$$

The equations, (9) – (15), get converted to the following new equations, (9A) – (15A) respectively, by substituting the values of b_4, c_4, and b_3 from (28), as well as by eliminating c_0 and c_1 (through the use of (17) and (22)).

$$2b_2+a_8\left(\frac{a_7}{2}+c_3\right)=a_6 \qquad (9A)$$

$$2b_1+a_8(b_2+c_2)+\frac{a_7^2}{4}-c_3^2=a_5 \qquad (10A)$$

$$2b_0+a_7b_2-2c_2c_3=a_4 \qquad (11A)$$

$$b_2^2+2a_8b_0+b_1(a_7+2c_3)-c_2^2=a_3 \qquad (12A)$$

$$b_0(a_7 - 2c_3) + 2b_1(b_2 + c_2) = a_2 \qquad (13A)$$

$$2b_0(b_2 - c_2) = a_1 \qquad (14A)$$

$$4b_0b_1 = a_0 \qquad (15A)$$

The supplementary equation (26) also gets modified as shown below:

$$a_7 + 2c_3 = \tfrac{4b_0}{b_2+c_2}. \qquad (26A)$$

Using (15A), b_0 is eliminated from the above set of equations, (11A) – (14A) and (26A), to obtain a new set of equations, (11B) – (14B) and (26B), respectively, as shown below:

$$\tfrac{a_0}{2b_1} + a_7b_2 - 2c_2c_3 = a_4 \qquad (11B)$$

$$b_2^2 - c_2^2 + \tfrac{a_0a_8}{2b_1} + b_1(a_7 + 2c_3) = a_3 \qquad (12B)$$

$$\tfrac{a_0}{4b_1}(a_7 - 2c_3) + 2b_1(b_2 + c_2) = a_2 \qquad (13B)$$

$$b_2 - c_2 = \tfrac{2a_1}{a_0}b_1 \qquad (14B)$$

$$b_2 + c_2 = \tfrac{a_0}{b_1(a_7+2c_3)}. \qquad (26B)$$

Using equations (14B) and (26B) we obtain the expressions for b_2 and c_2 as follows:

$$b_2 = \tfrac{a_0}{2b_1(a_7+2c_3)} + \tfrac{a_1b_1}{a_0} \qquad (14C)$$

$$c_2 = \tfrac{a_0}{2b_1(a_7+2c_3)} - \tfrac{a_1b_1}{a_0}. \qquad (26C)$$

Now b_2 and c_2 are eliminated from (9A), (10A), (11B), (12B), and (13B) using expressions (14C) and (26C), resulting in the following new equations, (9B), (10B), (11C), (12C), and (13C), respectively, in the unknowns b_1 and c_3 :

$$4a_1b_1^2(a_7 + 2c_3) = 2a_0a_6b_1(a_7 + 2c_3) - a_0a_8b_1(a_7 + 2c_3)^2 - 2a_0^2$$

(9B)

$$8b_1^2(a_7+2c_3) = 4a_5b_1(a_7+2c_3) - 2a_7b_1(a_7+2c_3)^2$$

$$+b_1(a_7+2c_3)^3 - 4a_0a_8$$

(10B)

$$a_1b_1^2(a_7+2c_3)^2 = a_0a_4b_1(a_7+2c_3) - a_0^2a_7$$

(11C)

$$2b_1^2(a_7+2c_3)^2 = 2a_3b_1(a_7+2c_3)$$

$$-4a_1b_1 - a_0a_8(a_7+2c_3)$$

(12C)

$$b_1 = \frac{2a_0a_7(a_7+2c_3) - a_0(a_7+2c_3)^2}{4a_2(a_7+2c_3) - 8a_0}.$$

(13C)

Notice that the expression (a_7+2c_3) appears frequently in the above equations. Therefore we make following substitution for the expression (a_7+2c_3) in these equations to simplify the algebraic manipulations:

$$d = a_7 + 2c_3. \qquad (29)$$

Hence the new equations obtained are as follows:

$$4a_1b_1^2d = 2a_0a_6b_1d - a_0a_8b_1d^2 - 2a_0^2 \qquad (9C)$$

$$8b_1^2d = 4a_5b_1d - 2a_7b_1d^2 + b_1d^3 - 4a_0a_8 \qquad (10C)$$

$$a_1b_1^2d^2 = a_0a_4b_1d - a_0^2a_7 \qquad (11D)$$

$$2b_1^2d^2 = 2a_3b_1d - 4a_1b_1 - a_0a_8d \qquad (12D)$$

$$b_1 = \frac{2a_0a_7d - a_0d^2}{4a_2d - 8a_0}. \qquad (13D)$$

Notice that there are five equations, (9C), (10C), (11D), (12D), and (13D), and two unknowns (b_1 and d) to be determined from these equations. While two equations are enough to determine the two unknowns, the remaining three equations are used to derive the conditions for the coefficients of the given nonic equation (1) to satisfy, so that the nonic is solvable. One may be tempted to eliminate b_1, using equation (13D), from the remaining equations (9C), (10C), (11D), and (12D); however a better way is to eliminate b_1^2, using (11D), from the remaining equations as described below.

Eliminating b_1^2 from (9C), using (11D) and rearranging, we obtain the following expression:

$$b_1 d(2a_6 d - a_8 d^2 - 4a_4) = (2a_0 d - 4a_0 a_7). \qquad \text{(9D)}$$

Similarly, we eliminate b_1^2 from (10C), using (11D) and rearranging, to get an expression as shown below:

$$b_1 d(4a_1 a_5 d - 2a_1 a_7 d^2 + a_1 d^3 - 8a_0 a_4) = 4a_0(a_1 a_8 d - 2a_0 a_7). \qquad \text{(10D)}$$

When b_1^2 is eliminated from (12D), using (11D) and rearranging, the following expression results:

$$2b_1[(a_1 a_3 - a_0 a_4)d - 2a_1^2] = a_0(a_1 a_8 d - 2a_0 a_7). \qquad \text{(12E)}$$

The latest equations to be considered for the elimination process are (9D), (10D), (11D), (12E), and (13D). Notice that the expression $(a_0 d - 2a_0 a_7)$ appears in both the equations, (9D) and (13D), and therefore it becomes easier to eliminate b_1 from (9D) using (13D); the resulting new expression (9E) obtained is as shown below:

$$d^4 - \tfrac{2a_6}{a_8} d^3 + \tfrac{4a_4}{a_8} d^2 - \tfrac{8a_2}{a_8} d + \tfrac{16a_0}{a_8} = 0. \qquad \text{(9E)}$$

Similarly the expression $(a_1 a_8 d - 2a_0 a_7)$ appears in equations, (10D) and (12E), and we eliminate b_1 from (10D) using (12E), to obtain the following expression:

$$d^4 - 2a_7 d^3 + 4a_5 d^2 - 8a_3 d + 16a_1 = 0. \qquad \text{(10E)}$$

We notice that the expression (12E) contains only linear terms of d in the numerator and the denominator, hence (12E) is used to eliminate b_1 from the remaining equations, (11D) and (13D), to obtain the new equations, (11E) and (13E), respectively as shown below:

$$d^4 + a_9 d^3 + a_{10} d^2 + a_{11} d + a_{12} = 0, \qquad \text{(11E)}$$

where a_9, a_{10}, a_{11}, and a_{12} are given by:

$$a_9 = \frac{2(a_0 a_4^2 - a_1 a_3 a_4 - 2a_0 a_1 a_7)}{a_1^2 a_8}$$

$$a_{10} = \frac{4(a_0^2 a_7^2 + a_1^2 a_4 a_8 + a_1 a_3^2 a_7 - a_0 a_3 a_4 a_7)}{a_1^2 a_8^2}$$

$$a_{11} = \frac{8a_7(a_0 a_4 - 2a_1 a_3)}{a_1 a_8^2}$$

$$a_{12} = \frac{16 a_1 a_7}{a_8^2}, \qquad (30)$$

and

$$d^3 + a_{13} d^2 + a_{14} d + a_{15} = 0, \qquad \text{(13E)}$$

where a_{13}, a_{14}, and a_{15} are as given below:

$$a_{13} = \frac{2(a_1 a_2 a_8 - a_1^2 + a_0 a_4 a_7 - a_1 a_3 a_7)}{a_1 a_3 - a_0 a_4}$$

$$a_{14} = \frac{4(a_1^2 a_7 - a_0 a_1 a_8 - a_0 a_2 a_7)}{a_1 a_3 - a_0 a_4}$$

$$a_{15} = \frac{8a_0^2 a_7}{a_1 a_3 - a_0 a_4}. \qquad (31)$$

There are four equations in d [(9E), (10E), (11E), and (13E)] and we can use any one of them to determine d. The remaining three equations serve as the conditions to be satisfied by the coefficients of the given nonic equation (1), so that it can be solved. For example if we choose the quartic equation (11E) to determine d, we obtain four values of d, namely, d_1, d_2, d_3, and d_4, and the desired value of d is the one that satisfies the remaining three equations. For example if d_3 satisfies the equations, (9E), (10E), and (13E), then d_3 is the desired value of d, and it is used to evaluate the unknown b_1 through the expression (12E) as given below:

$$b_1 = \frac{a_0(a_1 a_8 d_3 - 2a_0 a_7)}{2[(a_1 a_3 - a_0 a_4)d_3 - 2a_1^2]}. \qquad (12F)$$

Similarly the unknown c_3 is determined from (29) using d_3. The unknowns, b_2 and c_2 are determined from (14C) and (26C) respectively. Thus the given nonic equation (1) is successfully factored as shown below:

$$[x^4 + a_8 x^3 + (\tfrac{a_7}{2} - c_3)x^2 + \tfrac{2a_1}{a_0} b_1 x + 2b_1][x^2 + \tfrac{a_7}{2} + c_3](x^3 + b_2 + c_2) = 0.$$

Equating each of the factors in the above equation to zero, we obtain the following three equations:

$$x^4 + a_8 x^3 + \left(\tfrac{a_7}{2} - c_3\right) x^2 + \tfrac{2a_1}{a_0} b_1 x + 2b_1 = 0 \qquad (32)$$

$$x^2 + \tfrac{a_7}{2} + c_3 = 0 \qquad (33)$$

$$x^3 + b_2 + c_2 = 0. \qquad (34)$$

Solving the quartic equation (32), the four roots, x_1, x_2, x_3, and x_4, of nonic equation (1) are determined. Solving the quadratic equation (33), we establish the values of the two roots, x_5 and x_6, of the nonic equation. Solving the cubic (34), the remaining three roots, x_7, x_8, and x_9, of the nonic are determined. In the forthcoming sections, the behavior of the roots of such a solvable nonic equation, and the three conditions to be satisfied by its coefficients, are discussed. In the last section, a numerical example of the nonic equation, solvable through this method, is given.

Behavior of Roots

Observing the quadratic equation (33), we note that the x term is missing, which means the sum of the roots, x_5 and x_6, is zero as shown below:

$$x_5 + x_6 = 0,$$

or. equivalently, that one of the roots x_6 can be expressed in terms of the other root x_5 as:

$$x_6 = -x_5, \qquad (35)$$

where x_5 is given by:

$$x_5 = \left[-\left(\frac{a_7}{2} + c_3\right)\right]^{\frac{1}{2}}.$$

Consider the cubic equation (34); wherein the x^2 term and x term are missing. This means the roots of cubic (34), x_7, x_8, and x_9, are related as indicated below:

$$x_7 + x_8 + x_9 = 0$$

$$x_7(x_8 + x_9) + x_8 x_9 = 0. \qquad (36)$$

It can be easily shown from the above two equations (36) that the roots, x_8 and x_9, can be expressed in terms of the root x_7 as given below:

$$x_8 = \tfrac{x_7}{2}\left(-1 + 3^{\frac{1}{2}} i\right)$$

$$x_9 = \tfrac{x_7}{2}\left(-1 - 3^{\frac{1}{2}} i\right), \qquad (37)$$

where x_7 is the principal cube root of the cubic (34) and is given by:

$$x_7 = \left[-(b_2 + c_2)\right]^{\frac{1}{3}}.$$

Thus the given nonic equation (1) is solvable through the proposed method, if three of its roots are dependent as expressed in (35) and (37).

Conditions for Coefficients

Even though any one of the four equations, (9E), (10E), (11E), and (13E), can be chosen to determine d, there is some merit in choosing (11E) to determine d and choosing the other three equations to serve as conditions for the coefficients of (1). Notice that the coefficients, a_2, a_5, and a_6, are missing from (11E). However these coefficients (a_2, a_5, and a_6) are available in linear form in equations (13E), (10E), and (9E) respectively. Therefore, while the quartic equation (11E) is used to determine d (obtaining the four values, d_1, d_2, d_3, and d_4), the equations, (13E), (10E), and (9E), are used to obtain expressions for a_2, a_5, and a_6, respectively, as shown below, ultimately to serve as the conditions for the coefficients.

$$a_2 = \frac{(a_0a_4 - a_1a_3)d^3 + [2a_1^2 + 2a_7(a_1a_3 - a_0a_4)]d^2 + (4a_0a_1a_8 - 4a_1^2a_7)d - 8a_0^2a_7}{2d(a_1a_8d - 2a_0a_7)}$$

(13F)

$$a_5 = \frac{2a_7d^3 - d^4 + 8a_3d - 16a_1}{4d^2} \qquad (10F)$$

$$a_6 = \frac{a_8d^4 + 4a_4d^2 - 8a_2d + 16a_0}{2d^3} \qquad (9F)$$

If the given nonic equation (1) can be solved through the proposed technique, then at least one of the four values of d should yield the correct values of the coefficients, a_2, a_5, and a_6, from the expressions, (13F), (10F), and (9F), respectively.

Numerical Example

Let us solve the following nonic equation using the proposed technique:

$$x^9 + 2x^8 - 56x^7 - 170x^6 + 551x^5 + 1888x^4$$

$$+256x^3 + 4280x^2 + 18688x + 13440 \quad = 0.$$

First we must check whether the above nonic is solvable with this technique by applying the conditions, (9F), (10F), and (13F), for its coefficients. For this purpose we determine the parameters $a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}$, and a_{15}, using the expression sets, (30) and (31), as:

$a_9 = 191.8608,$ $a_{10} = 6243.216,$ $a_{11} = -94730.52,$

$a_{12} = -4186112,$ $a_{13} = 130.3843,$ $a_{14} = 3271.115,$

and $a_{15} = 3930.137.$

Using the above parameters the four values of d (d_1, d_2, d_3, and d_4) are determined by solving the quartic equation (11E), as indicated below:

$d_1 = -32,$ $d_2 = -146,$ $d_3 = -37.655455,$

and $d_4 = 23.794693.$

We notice that d_1 satisfies the equations, (9F), (10F), and (13F), confirming that the given nonic (in the example) is solvable through this method. Therefore d_1 becomes the desired value of d, and using d_1, the unknowns, b_1 and c_3 are evaluated from (12F) and (29) respectively as: $b_1 = -52.5$, $c_3 = 12$. Further, the unknowns, b_2 and c_2, are determined from (14C) and (26C) as: $b_2 = -69$, and $c_2 = 77$. Substituting these values, the quartic equation (32), the quadratic equation (33), and the cubic equation (34) are expressed as follows:

$$x^4 + 2x^3 - 40x^2 - 146x - 105 = 0$$

$$x^2 - 16 = 0$$

$$x^3 + 8 = 0.$$

Solving the above equations, the nine roots of the nonic equation (in the example) are obtained as: $-1, -3, -5, 7, 4, -4, -2, 1 - 3^{\frac{1}{2}}i$, and $1 + 3^{\frac{1}{2}}i$, where $i = \sqrt{-1}$.

Notice that in the numerical example above, the quartic equation (11E) yielded four values of d: $-32, -146, -37.655455$, and 23.794693. The coefficients of the nonic equation satisfied the conditions stipulated by (9F), (10F), and (13F), with $d = -32$. If the other three values of d are chosen and used in (9F), (10F), and (13F), we obtain three sets of a_6, a_5, and a_2, as indicated below.

For $d = -146$, the coefficients are:

$$a_2 = -10112.11, \qquad a_5 = -1248.014 \qquad \text{and} \qquad a_6 = -172.701.$$

For $d = -37.655455$, the coefficients are:

$$a_2 = -29908.3933 \qquad a_5 = 633.5535 \qquad \text{and} \qquad a_6 = -152.0208.$$

For $d = 23.794693$, the coefficients are:

$$a_2 = 34826.3913 \qquad a_5 = -918.3078 \qquad \text{and} \qquad a_6 = 160.229.$$

This means that for a set of six coefficients, a_0, a_1, a_3, a_4, a_7, and a_8, one can form four different nonic equations (with four sets of a_2, a_5,and a_6), which satisfy the conditions (9F), (10F), and (13F). One of the four nonic equations is solved here, and the other three are shown below:

$$x^9 + 2x^8 - 56x^7 - 172.701x^6 - 1248.014x^5 + 1888x^4$$
$$+256x^3 - 10112.11x^2 + 18688x + 13440 \quad = 0$$

$$x^9 + 2x^8 - 56x^7 - 152.0208x^6 + 633.5535x^5 + 1888x^4$$
$$+256x^3 + -29908.3933x^2 + 18688x + 13440 \quad = 0$$
$$x^9 + 2x^8 - 56x^7 + 160.229x^6 - 918.3078x^5 + 1888x^4$$
$$+256x^3 + 34826.3913x^2 + 18688x + 13440 \quad = 0.$$

The interested reader may find the roots of the above three nonic equations.

Conclusions

A decomposition technique to solve certain types of nonic equations is presented, wherein the given nonic equation is factored into quartic and quintic polynomials in a novel fashion. All nine roots of the nonic equation are determined in radicals by using the so-called supplementary equations.

Acknowledgements

The author thanks the management of Bharat Electronics for supporting this work.

References

[1] B. R. King, *Beyond the Quartic Equation,* Birkhauser, Boston, 1996.
[2] Eric W. Weisstein, Sextic Equation, *MathWorld—A Wolfram Web Resource,* http://mathworld.wolfram.com/SexticEquation.html.
[3] R. G. Kulkarni, Unified Method for Solving General Polynomial Equations of Degree Less than Five, *Alabama Journal of Mathematics,* Vol. 30, No. 1 & 2, Spring & Fall 2006, pp. 1 – 18.
[4] R. G. Kulkarni, On the Solution to Octic Equations, *The Montana Mathematics Enthusiast,* Vol. 4, No. 2, June 2007, pp. 193 – 209.

Deputy General Manager, HMC Division
Bharat Electronics Ltd., Jalahalli Post,
Bangalore-560013, INDIA.
Phone: +91-80-22195270, Fax: +91-80-28382738
e-mail: rgkulkarni@ieee.org

ACTM Fall Forum 2009

Exploring Math from Many Angles

October 15-16

Auburn University Montgomery

Thursday, October 15 - Starting at 1 pm (Registration Begins at 12)

For more information go to http://www.alabamamath.org

Speaker proposal forms are also available at the address above.