Fraction Division: We Can All Get It!!

By Dr. Cheryl Halcrow

Even now, when I think back on the first time that I taught Math for Elementary School Teachers, it is hard for me to believe that a person with a doctorate in education and bachelor's and master's degrees in mathematics could have been so confused when trying to *understand* division involving fractions — much less trying to *teach* it. I had never really thought about the meaning or nature of fraction division, and I am afraid my students may not have learned as much as they should have that semester.

The essence of the NCTM Standards is that mathematics should make sense to students (2000). When it does not, they revert to memorizing techniques and processes that are meaningless to them. This, of course, can lead to confusion over what techniques to use for which situations (e.g. when to find common denominators or when to work straight across when operating with rational numbers) and also to the inability to translate the algorithms to more abstract problems in higher mathematics. Bassarear (2008) states "It is simply not enough for an elementary teacher to know how to compute. It is crucial that the teacher also knows the *whys* behind the *hows"* (p. 284). Ma (1999) has the following to say about fraction division: "Division is the most complicated of the four operations. Fractions are often considered the most complex numbers in elementary school mathematics. Division by fractions, the most complicated operation with the most complex numbers, can be considered as a topic at the summit of arithmetic" (p.55). No wonder this combination can be daunting to anyone, even a professor!

Since that dismal semester, I have spent a great deal of time thinking about the whys of fraction division and how to teach these ideas. This article reflects some of the outcomes of that thought and gives a process or method to understand the essence of division involving fractions on a fairly deep level. It is not meant to be all encompassing on the subject, but it is my hope that it can give insight to both pre-service and practicing teachers and move them to further contemplate the most effective ways of teaching their own students.

The progressive development of the topic in this article mirrors, step by step, the development of the topic as I present it in the college classroom. It is not a concrete set of lessons for a classroom but rather a framework for truly understanding what happens when dividing by, or into, a fraction — whether the other number is a natural number or another fraction. The framework begins with understanding the two interpretations of natural number division and then building on that foundation to gradually incorporate fractions. Complete understanding of natural number division is essential and fundamental to understanding fraction division.

Natural Number Division

Putting differences in terminology aside, most authors of modern textbooks for pre-service elementary math courses give two distinct interpretations of natural number division. This splitting of division into its two essential meanings goes a long way to helping alleviate the confusion and lack of understanding of what division means among teachers and ultimately among their students. Beckmann (2005) gives the two distinct ways to interpret division as "how many groups?" and "how many in each group?", while Bassarear (2008) uses the terms "repeated subtraction model" and "partitioning model" for the same things, respectively. Musser, Burger, and Peterson (2006) use the terms "measurement division" (measuring out a specified number of objects to be in each group) and "partitive division" (dividing or partitioning a certain number of objects into a specified number of groups) for the two meanings. I have also seen the terms "subtractive division" and "sharing division." In my classroom, I have gravitated to using "repeated subtraction division" and "sharing or partitive division," as they seem to be fairly easily understood by my students. I also like "repeated subtraction" because it connects this interpretation of division to the "repeated addition" interpretation of multiplication. I will use these terms for the remainder of this article.

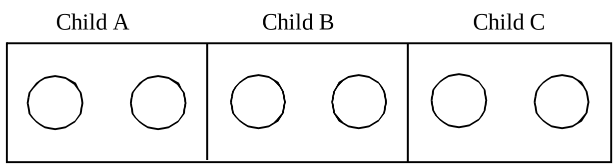
Consider the problem: $6 \div 2 = 3$. I generally begin by giving it the *repeated subtraction* interpretation (how many groups) and putting it into context for a more meaningful approach: 6 individual-size pizzas are given out so that each child will get 2. How many children will get pizza? $[6 \div 2 = 3]$. I then explain that this is actually the same problem/situation as: 6 pizzas are to be shared among 3 children. How many pizzas will each child get?

$[6 \div \ 3 = 2]$. The interpretation of the latter problem, however, is *sharing* division (how many in each group). See A and B below:

A. *Repeated Subtraction Division:*
$[6 \div 2 = 3]$ or [6 pizzas / 2 $\frac{\text{pizzas}}{\text{child}}$ = 3 children]

B. *Sharing Division:*
$[6 \div 3 = 2]$ or [6 pizzas / 3 children = 2 $\frac{\text{pizzas}}{\text{child}}$]

Even though these are two distinct interpretations of division, both interpretations can use the same drawing as a solution:

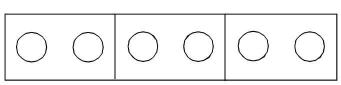


It is important to notice that even though the same drawing can serve as a visual model for the two interpretations of a given problem, the interpretations are labeled differently. This should be emphasized, because it will be important later when making sense of fraction division. In (A), using *repeated subtraction,* the divisor has a label which is the quotient of the labels of the dividend and the quotient, and the answer is given in the form "how many groups." In (B), *sharing division,* the denominator is labeled in terms of how many groups the numerator will be partitioned into. The answer has a label which is the quotient of the labels of the numerator and denominator and is interpreted as "how many in each group."

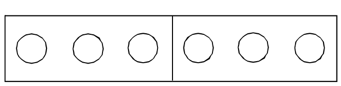
At this point I like to show my students, as an example, the difference between the visual models to $[6 \div 2 = 3]$ and $[6 \div 3 = 2]$, when both are given the same interpretation.

Repeated Subtraction
(how many groups)

$6 \div 2 = 3$

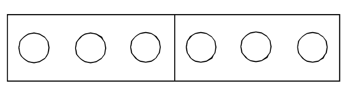


$6 \div 3 = 2$

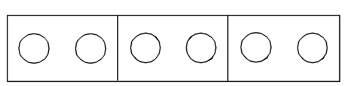


Sharing Division
(how many in each group)

$6 \div 2 = 3$



$6 \div 3 = 2$



Division by a Fraction

After doing this with several examples and varieties of contextual situations, the students feel quite comfortable with the two interpretations of division and will be ready for the introduction of fractions when the time arrives. I usually begin with one fraction and one natural number, and I try to give both interpretations of division. For example:

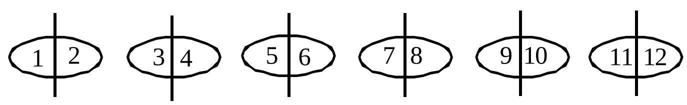
A. *Repeated Subtraction Division:* 6 pizzas are to be given out, $\frac{1}{2}$ pizza at a time. How many children will get pizza?

$$\left[6 \div \ \tfrac{1}{2} = 12\right] \text{ or } \left[6 \text{ pizzas } / \ \tfrac{1}{2}\ \tfrac{\text{pizza}}{\text{child}} = 12 \text{ children}\right].$$

B. *Sharing Division:* 6 pizzas will be shared by 12 children. How much pizza will each child get?

$$\left[6 \div \ 12 = \tfrac{1}{2}\right] \text{ or } \left[6 \text{ pizzas } / \ 12 \text{ children} = \tfrac{1}{2}\ \tfrac{\text{pizza}}{\text{child}}\right].$$

Both interpretations of this situation can use the same drawing as a visual solution:



As another example consider:

A. *Repeated Subtraction Division:* I am planning to run a two-mile race. There are water stops every $\frac{1}{4}$ mile, including the end of the race. How many water stops are there?

$$\left[2 \div \ \tfrac{1}{4} = 8\right] \text{ or } \left[2 \text{ miles } / \ \tfrac{1}{4}\ \tfrac{\text{miles}}{\text{water stop}} = 8 \text{ water stops}\right].$$

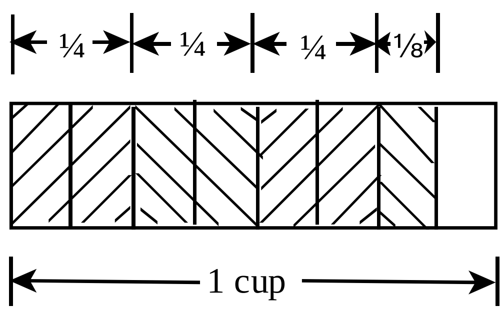
B. *Sharing Division:* I am planning to run a two-mile race. There are 8 water stops, including the end of the race. How far is it between each water stop?

$$\left[2 \div \ 8 = \tfrac{1}{4}\right] \text{ or } \left[2 \text{ miles } / \ 8 \text{ water stops} = \tfrac{1}{4}\ \tfrac{\text{miles}}{\text{water stop}}\right]$$

Ma (1999) says that American teachers tend to use mainly food items (pizza and candy bars) in their fraction division story problems, while Chinese teachers use many situations from normal life. Children can relate to pizza and candy and it is a nice segue from natural number division to division involving fractions, but it is important to broaden their contextual understanding of fraction division. I try to incorporate other types of story problems as I move into helping students understand what happens when one fraction is divided by another.

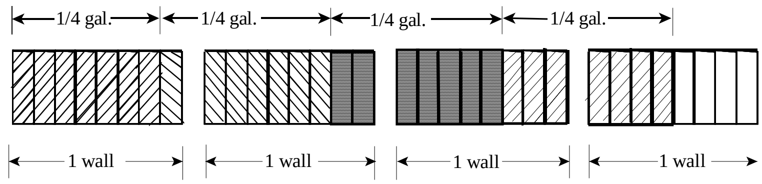
Division of One Fraction by Another

I generally split fraction division into two types, a fraction divided by a smaller fraction and a fraction divided by a larger fraction. The easier of the two to begin with is dividing a fraction by a smaller fraction. I might give my students a problem such as: You are making a recipe that calls for 7/8 cup of water, but you only have a 1/4 cup measure. How many times should you fill it to get 7/8 cup? I ask them first to tell me which interpretation of division this problem belongs to and then to draw a diagram to find a solution. I hope they will come up with a drawing that looks something like:



Finally, I ask them to give the division statement that solves this problem: $\left[\frac{7}{8}\text{ cup } / \ \frac{1}{4}\ \frac{\text{cup}}{\text{time}} = 3\frac{1}{2}\text{ times}\right]$. One can see by the way the labels are used that this is repeated subtraction division. Not all problems make sense using either interpretation, as do the preceding problems. However, beginning with problems such as these helps students to clearly see the difference between the two interpretations.

The same problem can be given a different contextual situation for a partitioning division interpretation: If 7/8 of a wall can be covered by 1/4 gallon of paint, how many walls can be covered by 1 gallon? First, I ask the students to try to find a solution by drawing a diagram. Then I ask them for the division statement that models the problem: $\left[\frac{7}{8}\text{ wall } / \ \frac{1}{4}\text{ gallon} = 3\frac{1}{2}\ \frac{\text{walls}}{\text{gallon}}\right]$. Note again that the proper use of labels helps students to see the correct interpretation of division and to understand why the answer makes sense. Students can be shown that ultimately they had to multiply 7/8 by 4 to get the answer, which is actually the "invert and multiply" strategy that we want them to understand. A sketch of the solution to this problem might look like:

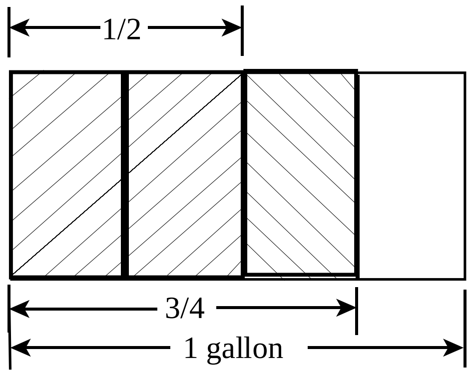


Consider the following problem from Beckmann (2005, p. 290): "A road crew is building a road. So far, 2/3 of the road has been completed and this portion of the road is 3/4 of a mile long. How long will the road be when it is completed?" Can you see that it is a fraction division problem? Most students who are not given the tools to understand fraction division would simply take the two fractions and randomly guess an operation to get an answer (Bassarear, 2008). If division jumps out at you, do you know which fraction to divide by the other? You might start by realizing that the answer will have to be of the form "miles per one road." This means you should use partitioning division as follows: $\left[\frac{3}{4}\text{ mile } / \ \frac{2}{3}\text{ road} = 1\frac{1}{8}\ \frac{\text{miles}}{\text{road}}\right]$.

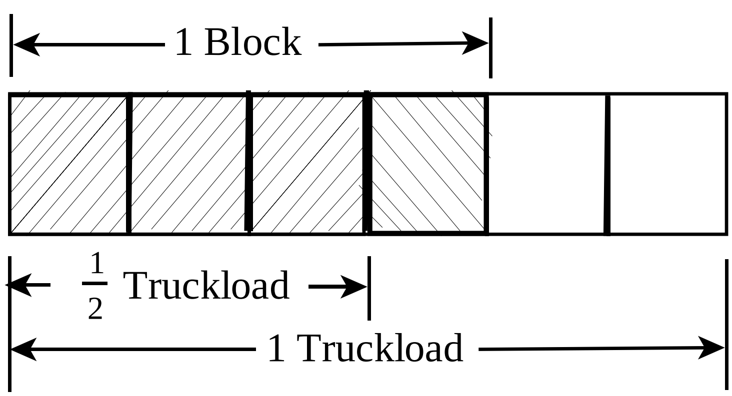
Finally, my students and I get to the most difficult type of fraction division, a smaller fraction divided by a larger fraction. It is important to again use both interpretations of division, put them into context, and have students attempt to draw diagrams for solutions and make sense of the answers. At some point along this journey, I ask my students to try to come up with their own story problems for a given interpretation of fraction division. For this last area I might say, "Write a repeated subtraction, or 'how many groups,' problem for $\frac{1}{2} \div \frac{3}{4}$; then draw a visual solution and give the answer with a proper label." I also give the same type of questions on exams. When students are able to do problems such as these correctly, I feel that they are on their way to becoming good elementary school math teachers.

After some time, students learn to think of this problem in terms of "how many 3/4ths are there in 1/2?" This can be helpful in finding both a contextual situation and its solution. The following could be an answer to the problem: The buckets that are usually used for picking berries are 3/4 gallon size. There are no more left, so I have to use my own bucket which is only 1/2 gallon size. What portion of berries will I get compared to others who are picking berries? $\left[\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}\right]$ or $\left[\frac{1}{2}\text{ gallon} / \ \frac{3}{4}\ \frac{\text{gallons}}{\text{portion}} = \frac{2}{3}\text{ portion}\right]$.

A drawing could also show this solution (there are 2/3 of 3/4 in 1/2):



The last example that I give is a partitive division example of a smaller fraction divided by a larger fraction. Consider, again, the problem $\frac{1}{2} \div \frac{3}{4}$. Could you write a sharing or partitioning division story for this problem and explain why it makes sense to solve it by "inverting and multiplying?" One possible answer is: It takes 1/2 of a truckload of cement to make a sidewalk that is 3/4 block long. What part of a truckload does it take to make a sidewalk that is 1 block long? $\left[\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}\right]$ or $\left[\frac{1}{2} \text{ truckload } / \ \frac{3}{4} \text{ block} = \frac{2}{3}\ \frac{\text{truckload}}{\text{block}}\right]$. A drawing for this solution might look like:



Using the invert and multiply strategy in this situation means that we would multiply 1/2 by 4/3. This is the same as first dividing 1/2 by 3 (which is 1/6) and then multiplying the answer by 4, resulting in 4/6 or 2/3. The 1/6 represents the portion of the

1/2 truckload that is used to make each 1/4 of the block. Since we want to know how much we need for 1 block, we have to take this 1/6 and multiply it by 4 since 1 block = 4/4 block.

What I have discussed is only the tip of the iceberg when it comes to strategies for understanding and making sense of fraction division. There are others that I have not even touched on. Almost all students can be taught to memorize the "invert and multiply" algorithm, but it is my goal to have this algorithm, along with the basic ideas and strategies of fraction division, make sense to students. In my classroom, we have only a number of days to cover this very broad topic, so it is my hope that this framework can give students the foundational understanding needed to dig a little deeper into the subject when they become practicing teachers. In turn, they will pass this understanding on to their students until, one day, all of us really will GET IT when it comes to dividing fractions!

References

[1] Bassarear, T. (2008). *Mathematics for Elementary Teachers,* fourth edition. Boston: Houghton Mifflin Company.

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[5] National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, Virginia.

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