Playing with the Twin Primes Conjecture and the Goldbach Conjecture

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Abstract. Here, we use only the simple part of the new method of induction, and we obtain a simple conjecture which is simultaneously stronger than the Goldbach Conjecture and the Twin Prime conjecture; and, using this simple conjecture, we explain why it is natural to conjecture that *the Twin Prime Conjecture can be seen as an obvious special case of the Goldbach Conjecture.*

Introduction

The Goldbach Conjecture (See [1], [2], [3], or [5]) states that every even integer $e \geq 4$ is of the form $e = p + p'$, where (p, p') is a pair of primes. Such being the case, we say that e is **goldbach,** if $e \geq 4$ is an even integer of the form $e = p + p'$, where (p, p') is a pair of primes. We say that an even integer integer $e \geq 4$ is **goldbachian** if every even integer υ with $4 \leq \upsilon \leq e$ is goldbach.

Observation: Let $n > 2$ be an integer. Then the following are equivalent:

(i) $2n + 2$ is goldbachian.

(ii) $2n$ is goldbachian and $2n + 2$ is goldbach.

We recall that an integer t is a **twin prime** (See [4], [5], or [6]), if $t \geq 3$ is prime and if $t-2$ or $t+2$ is also prime. For example: 1000000000061 and 1000000000063 are twin primes (See [5]). The Twin Prime Conjecture states that there are infinitely many twin primes.

For every integer $n \geq 2$, we define $G'(n)$, g'_n, $P(n)$, p_n, $T(n)$, and t_n as follows:

$$\begin{aligned}
G'(n) &= \{g' : 1 < g' \leq 2n, \text{ and } g' \text{ is goldbachian}\} \\
g'_n &= \max\nolimits_{g' \in G'(n)} g' \\
P(n) &= \{p : p \text{ is prime and } 1 < p < 2n\} \\
p_n &= \max\nolimits_{p \in P(n)} p \\
T(n) &= \{t : t \text{ is a twin prime and } 1 < t < 2n\} \\
t_n &= \max\nolimits_{t \in T(n)} t
\end{aligned}$$

Observation: Let $n \geq 2$ be an integer, and consider g'_{n+1}. We have the following three properties:

(i) $g'_{n+1} \leq 2n + 2$.
(ii) $g'_{n+1} < 2n + 2$ if and only if $g'_{n+1} = g'_n$.
(iii) $g'_{n+1} = 2n + 2$ if and only if $2n + 2$ is goldbachian.

Let **A** be the following assertion:

Assertion A: For every integer $m \geq 2$, the following two properties, w(A,m) and o(A,m), are equivalent:

w(A,m) : $2m + 2$ is Goldbach

o(A,m) : $t_m \cdot \sum_{t \in T(m)} t > p_m$.

Using only the simple part of the new method of induction, we prove a theorem which immediately implies the following result R:

Theorem R: Suppose that Assertion A holds. Then the Twin Prime Conjecture and the Goldbach Conjecture simultaneously hold.

Theorem R clearly says that Assertion A is stronger than either the Twin Prime Conjecture or the Goldbach Conjecture and, using the previous theorem, we explain why it is natural to conjecture that the Twin Prime Conjecture is only a special case of the Goldbach Conjecture.

1. The Proof of a Theorem Which Implies Theorem R

Before we state and prove our main theorem, we must introduce two simple definitions. First, let $n \geq 2$ be an integer. We say that $z(n)$ is a **cache** of n if $z(n)$ is an integer such that $0 \leq z(n) < n$. (For example, suppose that $n = 13$. Then $z(n)$ is a cache of n if and only if $z(n) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.) Second, for every pair of integers $(n, z(n))$ such that $n \geq 2$ and $0 \leq z(n) < n$, we define $z(n, 2)$ as $z(n, 2) \equiv z(n) \bmod 2$.

The following theorem immediately implies Theorem R, stated in the Introduction.

Theorem 1: Let $(n, z(n))$ be a pair of integers such that $n \geq 5$ and $z(n)$ is a cache of n. Suppose also that Assertion A holds. Then we have the following:

(1) If $z(n) \equiv 0 \bmod 2$, then $t_n \cdot \sum_{t \in T(n)} t > p_n - z(n)$

(2) If $z(n) \equiv 1 \bmod 2$, then $2n + 2$ is goldbachian.

Remark 1: If we suppose that Theorem 1 is false, then there exists a pair $(n, z(n))$ such that $(n, z(n))$ is a counter-example of Theorem 1 with n minimal and, given n, $z(n, 2)$ is minimal. A consequence of the existence of such a pair is that, by minimality of $z(n, 2)$, every pair $(n, f(n))$ such that $f(n)$ is a cache of n and $f(n, 2) < z(n, 2)$ is *not* a counter-example of Theorem 1.

To prove Theorem 1, we use the following lemma.

Lemma 1: Suppose that $n = 5$. Then Theorem 1 is satisfied.

Proof. Indeed, since $n = 5$, clearly $z(n) \in \{0, 1, 2, 3, 4\}$, and it suffices to show that Theorem 1 is satisfied for all $z(n) \in \{0, 1, 2, 3, 4\}$. So, we have to consider two cases: namely, the case where $z(n) \in \{0, 2, 4\}$, and case where $z(n) \in \{1, 3\}$.

Case 1: $z(n) \in \{0, 2, 4\}$. Clearly $z(n) \equiv 0 \bmod 2$ and we must show that property (1) of Theorem 1 holds. Recalling that $n = 5$, clearly $T(n) = \{3, 5, 7\}$, $t_n = p_n = 7$, and $t_n \cdot \sum_{t \in T(n)} t > p_n$. (In particular, $t_n \cdot \sum_{t \in T(n)} t > p_n - z(n)$); so property (1) of Theorem 1 holds, and Theorem 1 is satisfied.

Case 2: $z(n) \in \{1, 3\}$. Clearly $z(n) \equiv 1 \bmod 2$ and we must show that property (2) of Theorem 1 holds. Recalling that $n = 5$, we have $2n+2 = 12$ and $2n+2$ is goldbachian (because 12 is clearly goldbachian). Property (2) of Theorem 1 holds, and Theorem 1 is satisfied. $\square$

Proof of Theorem 1: Otherwise, let the pair $(n, z(n))$ be a counter-example such that n is least and $z(n, 2)$ is least (Such a pair exists by Remark 1). Then, we have the following observations:

Observation 1.1: Note that $n \geq 6$, p_n and p_{n-1} are odd, $p_n \leq 2n - 1$, and $2n$ is goldbachian.

To see this, note that $n \geq 6$ by using Lemma 1; so p_n and p_{n-1} are odd, and clearly $p_n \leq 2n - 1$. Now. to prove Observation 1.1, it suffices to show that $2n$ is goldbachian. For a fact, $2n$ is goldbachian. Consider the pair $(m, z(m))$ such that $m = n - 1$ and $z(m) = 1$; since $n \geq 6$ (by Lemma 1), clearly $m \geq 5$, $z(m)$ is a cache of m, and $m < n$. Then, by the minimality of n, the pair $(m, z(m))$ satisfies Theorem 1. Clearly $z(m) \equiv 1 \bmod 2$ and therefore, property (2) of Theorem 1 is satisfied by the pair $(m, z(m))$. So $2m+2$ is goldbachian, and recalling that $m = n-1$, clearly $2n$ is goldbachian. Observation 1.1 follows.

Observation 1.2: If $2n-1$ is not prime, then $t_n \cdot \sum_{t \in T(n)} t > p_n$.

Indeed, observing that p_{n-1} and p_n are odd, that $p_n \leq 2n-1$ (by Observation 1.1), and that $2n-1$ is not prime, clearly

$$p_n = p_{n-1}. \tag{1.1}$$

Now suppose that the pair $(m, z(m))$ is such that $m = n-1$ and $z(m) = 0$; since $n \geq 6$ (by Observation 1.1), clearly $m \geq 5$, $z(m)$ is a cache of m, and $m < n$. Then, by the minimality of n, the pair $(m, z(m))$ satisfies Theorem 1. Clearly $z(m) \equiv 0 \bmod 2$, and therefore property (1) of Theorem 1 is satisfied by the pair $(m, z(m))$. So

$$t_m \cdot \sum_{t \in T(m)} t > p_m - z(m). \tag{1.2}$$

Recalling that $z(m) = 0$ and $m = n-1$, and that $p_n = p_{n-1}$ (by equation (1.1)), inequality (1.2) becomes:

$$t_{n-1} \cdot \sum_{t \in T(n-1)} t > p_n. \tag{1.3}$$

Clearly $t_n \cdot \sum_{t \in T(n)} t \geq t_{n-1} \cdot \sum_{t \in T(n-1)} t$, and inequality (1.3) immediately implies that $t_n \cdot \sum_{t \in T(n)} t > p_n$. Observation 1.2 follows.

Observation 1.3: Let $(n, z(n))$ be given, where $(n, z(n))$ is the aforementioned counter-example of Theorem 1, with n least and $z(n, 2)$ least. Then $z(n) \equiv 0 \bmod 2$.

Otherwise,

$$z(n) \equiv 1 \bmod 2, \tag{1.4}$$

and we have the following claims.

Claim 1.1: $t_n \cdot \sum_{t \in T(n)} t > p_n$.

To see this, consider $(n, z(n))$ and look at $z(n, 2)$. Since $z(n) \equiv 1 \bmod 2$ (by congruence (1.4)), clearly $z(n, 2) = 1$. Now let the pair $(n, f(n))$ be such that $f(n) = 0$; observing that $n \geq 6$ (by Observation 1.1), then $f(n)$ is a cache of n with $f(n, 2) = 0$. Clearly $f(n, 2) < z(n, 2)$ (where $z(n)$ and $f(n)$ are two caches of n); then, by the minimality of $z(n, 2)$, the pair $(n, f(n))$ satisfies Theorem 1 (See Remark 1). Clearly $f(n) \equiv 0 \bmod 2$ and therefore, property (1) of Theorem 1 is satisfied by the pair $(n, f(n))$. So

$t_n \cdot \sum_{t \in T(n)} t > p_n - f(n)$, and clearly $t_n \cdot \sum_{t \in T(n)} t > p_n$, because $f(n) = 0$. Thus, we have established Claim 1.1.

Claim 1.2: $2n + 2$ is goldbach.

Indeed, since Assertion A holds, then, using Claim 1.1, it immediately follows that $2n + 2$ is goldbach. Claim 1.2 follows.

Claim 1.3: Property (2) of Theorem 1 is false.

This claim is immediate (Since $z(n) \equiv 1 \bmod 2$ (by congruence (1.4)), and since the pair $(n, z(n))$ is a counter-example of Theorem 1).

Claim 1.4: $2n + 2$ is not goldbachian.

Indeed, observing that property (2) of Theorem 1 is false (by Claim 1.3), it follows that $2n + 2$ is not goldbachian. Claim 1.4 follows.

These four claims having been made, observing that $2n + 2$ is goldbach (by Claim 1.2), and since $2n$ is goldbachian (by Observation 1.1), clearly $2n + 2$ is goldbachian, and this contradicts Claim 1.4. Thus, Observation 1.3 follows.

These simple three observations having been made, let the pair $(n, z(n))$ be the aforementioned minimal counter-example of Theorem 1). We observe the following consequences, for the sake of deriving a contradiction.

Consequence 1.1: Property (1) of Theorem 1 is false.

Indeed, this consequence is immediate, since $z(n) \equiv 0 \bmod 2$ (by Observation 1.3), and since the pair $(n, z(n))$ is a counter-example of the Theorem 1.

Consequence 1.2: $t_n \cdot \sum_{t \in T(n)} t \le p_n - z(n)$.

This consequence follows immediately from Consequence 1.1.

Consequence 1.3: $2n - 1$ is prime.

Otherwise, Observation 1.2 implies that $t_n \cdot \sum_{t \in T(n)} t > p_n$, and this contradicts Consequence 1.2, as $z(n) \ge 0$.

Consequence 1.4: $2n + 2$ is goldbach.

Indeed, observing that $2n - 1$ is prime (by Consequence 1.3), clearly $2n+2$ is goldbach (note that $2n+2 = 2n-1+3$, where 3 and $2n - 1$ are primes and $n \ge 6$ (by Observation 1.1)). Consequence 1.4 follows.

These four consequences having been observed, recalling that Assertion A holds, and since $2n + 2$ is goldbach (by Consequence 1.4), it immediately follows that $t_n \cdot \sum_{t \in T(n)} t > p_n$. This contradicts Consequence 1.2, because $z(n) \ge 0$. Thus, Theorem 1 follows.

Corollary 1: Suppose that Assertion A holds. Then, for every integer $n \geq 1$, $2n+2$ is goldbachian.

Proof. The proof is immediate if $n \in \{1, 2, 3, 4\}$. If $n \geq 5$, let the pair $(n, z(n))$ be such that $z(n) = 1$. The pair $(n, z(n))$ has the property that $0 \leq z(n) < n$, where $n \geq 5$, $z(n) = 1 \bmod 2$, and $z(n)$ is a cache of n. Then property (2) of Theorem 1 is satisfied by the pair $(n, z(n))$. Therefore $2n+2$ is goldbachian. □

Corollary 2: Suppose that Assertion A holds. Then, the Goldbach Conjecture holds.

Proof. Indeed, the Goldbach Conjecture is an immediate consequence of Corollary 1. □

Corollary 3: Suppose that Assertion A holds. Then, for every integer $n \geq 2$, we have $t_n \cdot \sum_{t \in T(n)} t > p_n$.

Proof. The proof is immediate if $n \in \{2, 3, 4\}$. If $n \geq 5$, let the pair $(n, z(n))$ be such that $z(n) = 0$. The pair $(n, z(n))$ has the property that $0 \leq z(n) < n$, where $n \geq 5$, $z(n) = 0 \bmod 2$, and $z(n)$ is a cache of n. Then property (1) of Theorem 1 is satisfied by the pair $(n, z(n))$. Therefore $t_n \cdot \sum_{t \in T(n)} t > p_n - z(n)$, and clearly $t_n \cdot \sum_{t \in T(n)} t > p_n$ (because $z(n) = 0$). □

Corollary 4: Suppose that Assertion A holds. Then, the Twin Prime Conjecture holds.

Proof. Indeed, the Twin Prime Conjecture is an immediate consequence of Corollary 3. □

Corollary 5: Suppose that Assertion A holds. Then, for every integer $n \geq 5$, $t_n \cdot \sum_{t \in T(n)} t > n$.

Proof. Observing that there exists a prime between n and $2n$ (for every integer $n \geq 1$), then, using definition of p_n and Corollary 3, we immediately deduce that $p_n \geq n$; so. $t_n \cdot \sum_{t \in T(n)} t > n$. □

Using Corollary 2 and Corollary 5. the following result becomes immediate.

Theorem R: Suppose that Assertion A holds. Then the Goldbach Conjecture and the Twin Prime Conjecture simultaneously hold.

Proof. Corollary 2 says that the Goldbach Conjecture holds, and Corollary 5 is stronger than the Twin Prime Conjecture. □

Conjecture 1: Assertion (A) holds. (Note that Conjecture 1 simultaneously implies the Goldbach Conjecture and the Twin Prime Conjecture, via Theorem R.)

Conclusion

It is natural to conjecture that the Twin Prime Conjecture can be seen as an obvious special case of the Goldbach Conjecture. Indeed, let A* be the following assertion:

Assertion A*: For every integer $m \geq 2$, the following two properties, w(A*,m) and o(A*,m), are such that o(A*,m) $\Rightarrow$ w(A*,m).

w(A*.m): $2m + 2$ is goldbach.

o(A*.m): $t_m \cdot \sum_{t \in T(m)} t > p_m$

Observe that Assertion A* is somewhat similar to Assertion A, which is stronger than the Goldbach Conjecture and the Twin Prime Conjecture, via Theorem R. More precisely, Assertion A clearly implies Assertion A*.

Conjecture 2: Assertion A and Assertion A* are equivalent.

Note that Conjecture 2 implies that the Twin Prime Conjecture is a special case of the Goldbach Conjecture.

Proof. Suppose that Conjecture 2 holds. If the Goldbach Conjecture holds, then clearly Assertion A* holds; so Assertion A holds (because Assertion A* and Assertion A are equivalent by hypothesis), and Theorem R implies that the Twin Prime Conjecture holds. □

Conjecture 3: Suppose that Assertion A* holds. Then, the Goldbach Conjecture and the Twin Prime Conjecture simultaneously hold.

Note that Conjecture 3 immediately implies that the Twin Prime Conjecture is a special case of the Goldbach Conjecture.

Proof. Suppose that Conjecture 3 holds. If the Goldbach Conjecture holds, then clearly Assertion A* holds, and as a consequence, the Twin Prime Conjecture holds. □

Now. using Theorem R, Assertion A, the previous two conjectures, and observing that there is not a great difference between Assertion A and A*, it becomes natural to conjecture that:

Conjecture 4: The Twin Prime Conjecture is only an obvious special case of the Goldbach conjecture.

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