Solutions and Discussions

Problem 1 — Volume 28, No. 1, Spring, 2004

Suppose that $ax^2 + bx + c = 0$ where a, b, c are odd integers. Prove that x cannot be a rational number.

Solution

Ricky Farr, University of South Carolina Upstate, Spartanburg, SC.

We begin by introducing two lemmas.

LEMMA 1. *For all integers a and b, ab is even iff a is even or b is even. (Equivalently, for all integers a and b, ab is odd iff a is odd and b is odd.)*

This is a well-known result from elementary number theory and the proof is left to the reader.

LEMMA 2. *For all integers j and k, $(j + k + 1)(j - k)$ is even.*

PROOF. Fix integers j and k. Note that since the integers are closed under addition and subtraction, $(j + k + 1)$ and $(j - k)$ are both integers. (Since it is well known that the integers are closed under addition, subtraction, and multiplication, we will draw upon this fact where necessary, without making further mention of it.)

CASE 1. *Assume that j is even and k is odd. Then there exist integers m and n satisfying the conditions, $j = 2m$ and $k = 2n+1$. Then $j+k+1 = (2m)+(2n+1)+1 = 2(m+n+1)$. Thus $j+k+1$ is even, whereby $(j + k + 1)(j - k)$ is even by Lemma 1.*

CASE 2. *Assume j is even and k is even. Then there exist integers m and n satisfying the conditions, $j = 2m$ and $k = 2n$. Then $j - k = 2m - 2n = 2(m - n)$, and $j - k$ is even. By Lemma 1, $(j + k + 1)(j - k)$ is even.*

CASE 3. *Assume that j is odd and k is even. Then there exist integers m and n satisfying the conditions, $j = 2m+1$ and $k = 2n$. Then $j+k+1 = (2m+1)+(2n)+1 = 2(m+n+1)$. Thus $j+k+1$ is even, whereby $(j+k+1)(j-k)$ is even by Lemma 1.*

CASE 4. *Assume that j is odd and k is odd. Then there exist integers m and n satisfying the conditions, $j = 2m+1$ and $k = 2n+1$. Then $j-k = (2m+1)-(2n+1) = 2m-2n = 2(m-n)$, and $j-k$ is even. By Lemma 1, $(j+k+1)(j-k)$ is even.*

□

We now prove the main result by contradiction:

Assume that a, b, and c, are odd integers and the equation $ax^2+bx+c=0$ has a rational root. It is well known that a quadratic equation with integer coefficients has a rational solution iff the discriminant is the square of a rational number. Thus, by our assumption, there exists an integer, say n, satisfying the condition $b^2-4ac=n^2$. Rearranging terms, we have $b^2-n^2=4ac$, or equivalently, $(b+n)(b-n)=4ac$. Clearly $4ac$ is even, hence $(b+n)(b-n)$ must also be even. Since b is an odd integer, there is an integer j satisfying the condition, $b=2j+1$.

CASE 1. Assume that n is even. Then there is an integer k satisfying the condition, $n=2k$. Then $(b+n)(b-n) = (2j+1+2k)(2j+1-2k) = (2(j+k)+1)(2(j-k)+1)$, which is odd, by Lemma 1, as it is the product of odd integers. However, this contradicts the fact that $(b+n)(b-n)$ must be even.

CASE 2. Assume that n is odd. Then there is an integer, say k, satisfying the condition, $n=2k+1$. Thus, $(b+n)(b-n) = [(2j+1)+(2k+1)][(2j+1)-(2k+1)] = 2(j+k+1)2(j-k) = 4(j+k+1)(j-k)$.

Recalling the equation $(b+n)(b-n)=4ac$ and substituting the above result yields $4(j+k+1)(j-k)=4ac$. Canceling the 4's yields $(j+k+1)(j-k)=ac$.

The left side is even by Lemma 2 and the right side is odd by Lemma 1 (recall a and c are odd integers) — a contradiction.

This concludes the argument.

Problem 1 — Volume 29, No. 1 & 2, Spring/Fall, 2005

You are organizing a new basketball league, and 50 people want to participate. How many distinct ways can you divide these 50 people into 10 teams with 5 players each?

Solution

Elliott Ventiere, Senior, Troy University, Troy, AL.

In order to find the total number of distinct ways the teams can be formed, we must first compute several combinations. Since the total number of people to select from is equal to 50 and the amount of players per team must be equal to 5, the first combination will be C_r^n or $\binom{n}{r}$ where $n = 50$ and $r = 5$.

$\binom{n}{r} = \binom{50}{5} = 2,118,760$

This is equivalent to the total number of ways that the first team can be selected. To find the next team, we must select the next 5 players from the remaining 45 people. Therefore, the next combination is C_r^n or $\binom{n}{r}$ where $n = 45$ and $r = 5$.

$\binom{n}{r} = \binom{45}{5} = 1,221,759$

We continue this pattern until all 10 teams are selected. The final combination will be C_5^5 or $\binom{5}{5}$. Since these combinations only represent the number of ways each individual team can be selected, we must find a way to compute the total number of distinct ways that all 10 teams can be formed. This is simply the product of the number of ways that each team can be formed (i.e., the product of the combinations):

$N = \binom{50}{5}\binom{45}{5}\binom{40}{5}\binom{35}{5}\binom{30}{5}\binom{25}{5}\binom{20}{5}\binom{15}{5}\binom{10}{5}\binom{5}{5} =$

$49,120,458,506,088,132,224,064,306,071,170,476,903,628,800.$

Problem 7 — Volume 27, No. 1 & 2, Spring/Fall, 2003

How many total gifts are given in the song "The 12 Days of Christmas"? If additional verses are added in the same pattern, how many total gifts would be given in the song "The N Days of Christmas"?

Solution

Jordan Medeiros, Senior, Troy University, Troy, AL.

The number of gifts given on the n^{th} day of Christmas is $(1 + 2 + 3 + \ldots + n) = \sum_{i=1}^{n} i$. Therefore, the number of gifts given in the song "The 12 Days of Christmas" is $\sum_{n=1}^{12} \left(\sum_{i=1}^{n} i\right) = 364$.

To generalize this result, note again that the number of gifts given on the n^{th} day of Christmas is $(1 + 2 + 3 + \ldots + n) = \sum_{i=1}^{n} i = t_n$, where t_n is the n^{th} triangular number. Therefore, the number of gifts given during N days of Christmas is given by the sum of the first N triangular numbers, $\sum_{n=1}^{N} t_n$. One thing that we know about triangular numbers is that the sum of any two successive triangular numbers is a square number. In particular, $t_n + t_{n+1} = s_{n+1}$, where s_{n+1} is the $(n+1)^{st}$ square number. With this motivation, we examine the first few sums:

$\sum_{n=1}^{1} t_n = 1^2$

$\sum_{n=1}^{2} t_n = (1+3) = 2^2$

$\sum_{n=1}^{3} t_n = 1 + (3+6) = 1^2 + 3^2$

$\sum_{n=1}^{4} t_n = (1+3) + (6+10) = 2^2 + 4^2$

$\sum_{n=1}^{5} t_n = 1 + (3+6) + (10+15) = 1^2 + 3^2 + 5^2$

$\sum_{n=1}^{6} t_n = (1+3) + (6+10) + (15+21) = 2^2 + 4^2 + 6^2$

etc.

We are looking for an expression for the sum of the first N triangular numbers that is independent of whether N is even or odd. Note that:

$$\begin{aligned}
2\sum_{n=1}^{N} t_n &= \sum_{n=1}^{N} t_n + \left(\sum_{n=1}^{N-1} t_n + t_N\right) \\
&= \left(1^2 + 2^2 + 3^2 + \ldots + N^2\right) + t_N \\
&= \sum_{n=1}^{N} n^2 + t_N \\
&= \sum_{n=1}^{N} n^2 + \sum_{n=1}^{N} n.
\end{aligned}$$

i.e., $2\sum_{n=1}^{N} t_n = \sum_{n=1}^{N} n^2 + \sum_{n=1}^{N} n$

It is well known that $\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$, and that $\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$. Thus,

$$\begin{aligned}
2\sum_{n=1}^{N} t_n &= \sum_{n=1}^{N} n^2 + \sum_{n=1}^{N} n \\
&= \frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2} \\
&= \frac{N(N+1)(N+2)}{3}.
\end{aligned}$$

i.e., $2\sum_{n=1}^{N} t_n = \frac{N(N+1)(N+2)}{3}.$

Therefore, $\sum_{n=1}^{N} t_n = \frac{N(N+1)(N+2)}{6}$.

In the song "The N Days of Christmas," there are $\frac{N(N+1)(N+2)}{6}$ total gifts are given.

Solutions, comments, and discussions should be sent to:

Ken Roblee
Department of Math & Physics
232 MSCX
Troy State University
Troy, AL 36082
(334)670-3406
FAX (334)670-3796
kroblee@troyst.edu

Vicky Eichelberg
Saint James School
6010 Vaughn Road
Montgomery, AL 36116
(334)277-8033
FAX (334)277-8059
eichelberg2@charter.net