Limits of Quadratic Roots

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Teachers of calculus are always on the lookout for settings in which prior algebraic topics can be revisited using more advanced analytical tools. The quadratic equation provides a good opportunity for such a study.

All calculus students are familiar with the two solutions of the quadratic equation. If $ax^2 + bx + c = 0$, the two solutions are $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. If $a = 0$, these solutions cannot be used since they involve division by 0. In this situation the equation becomes $bx + c = 0$, which has a simple solution, $x = -\frac{c}{b}$.

What happens to the two solutions of the quadratic equation if a is "close to," but not equal to, 0? In other words: "As $a \to 0$ while b and c are held constant, do the two solutions approach limits as well?" The evaluation of the limits of the two solutions follows. In this analysis, we assume that $a > 0$, since any quadratic equation with $a < 0$ can be re-written in equivalent form with $a > 0$. Also, for the sake of simplicity, we consider only the case in which $b > 0$.

Solution 1: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$$\begin{aligned}
\lim_{a\to 0} \frac{-b+\sqrt{b^2-4ac}}{2a} &= \lim_{a\to 0} \frac{\left(-b+\sqrt{b^2-4ac}\right)\left(-b-\sqrt{b^2-4ac}\right)}{2a\left(-b-\sqrt{b^2-4ac}\right)} \\
&= \lim_{a\to 0} \frac{b^2-\left(b^2-4ac\right)}{2a\left(-b-\sqrt{b^2-4ac}\right)} \\
&= \lim_{a\to 0} \frac{2c}{\left(-b-\sqrt{b^2-4ac}\right)} \\
&= \frac{2c}{-2b} = -\frac{c}{b}
\end{aligned}$$

Solution 2: $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

$$\begin{aligned}
\lim_{a\to 0^+} \tfrac{-b-\sqrt{b^2-4ac}}{2a} &= \lim_{a\to 0^+} \tfrac{\left(-b-\sqrt{b^2-4ac}\right)\left(-b+\sqrt{b^2-4ac}\right)}{2a\left(-b+\sqrt{b^2-4ac}\right)} \\
&= \lim_{a\to 0^+} \tfrac{b^2-\left(b^2-4ac\right)}{2a\left(-b+\sqrt{b^2-4ac}\right)} \\
&= \lim_{a\to 0^+} \tfrac{2c}{\left(-b+\sqrt{b^2-4ac}\right)} \\
&= ???
\end{aligned}$$

It appears that the limit above depends on the value of c. Certainly it merits further investigation.

$$\begin{aligned}
\text{If } c > 0, \quad \text{then} \quad & b^2 - 4ac < b^2 \\
\Rightarrow \quad & \sqrt{b^2-4ac} < \sqrt{b^2} = b. \\
\text{i.e.,} \quad & \sqrt{b^2-4ac} < b \\
\Rightarrow \quad & -b + \sqrt{b^2-4ac} < -b + b = 0. \\
\text{i.e.,} \quad & -b + \sqrt{b^2-4ac} < 0. \\
\Rightarrow \quad & \lim_{a\to 0^+} \left(-b + \sqrt{b^2-4ac}\right) = 0^-.
\end{aligned}$$

Similarly, for $c < 0$, we have: $\lim_{a\to 0^+} \left(-b + \sqrt{b^2-4ac}\right) = 0^+$. What this means, as far as our analysis is concerned, is that for $c \neq 0$:

$$\lim_{a\to 0} \frac{-b-\sqrt{b^2-4ac}}{2a} = \lim_{a\to 0} \frac{2c}{\left(-b+\sqrt{b^2-4ac}\right)} = -\infty.$$

Incidentally, for the case in which $b < 0$, we have:

Solution 1: $\lim_{a\to 0} \frac{-b+\sqrt{b^2-4ac}}{2a} = \infty$

Solution 2: $\lim_{a\to 0} \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{c}{b}$.

It is not surprising that as $a \to 0$, one of the solutions approaches $-\frac{c}{b}$; this is the value that the solution would be if, in fact, a were *equal* to 0. Many students reason that since only one solution emerges when $a = 0$, both of the limits of the separate solutions of the quadratic equation must converge to the solution of the case in which $a = 0$. This, however, is not the case. One of the solutions approaches either ∞ or $-\infty$ as a limit.

The quadratic equation below illustrates this pattern of limits:

$$0.00008x^2 + 7.32814x - 4.10182 = 0$$

Observe that the value of a is very small relative to those of b and c.

$$\text{The solutions are: } x = \frac{-7.32814 \pm \sqrt{(7.32814)^2 - 4(0.00008)(-4.10182)}}{2(0.00008)}$$

$$= \frac{-7.32814 \pm \sqrt{53.703126}}{0.00016}$$

$$= \frac{-7.32814 \pm 7.32823}{0.00016}$$

$$\Rightarrow x = 0.559732 \text{ or } x = -91602$$

It is clear that one solution, $x = 0.559732$, is approximately equal to $\frac{4.10182}{7.32814} = -\frac{c}{b}$. The magnitude of the other solution, $x = -91602$, is very large, suggesting that as $a \to 0$, the solution approaches $-\infty$.

Other Cases

Let us next examine the setting in which $b \to 0$, while a and c are held constant. Again, we assume that $a > 0$ and, in order to avoid an imaginary limit, we assume that $c < 0$.

Solution 1: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$$\lim_{b\to 0} \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{0+\sqrt{-4ac}}{2a} = \frac{\sqrt{-4ac}}{2a} = \frac{\sqrt{-c}}{\sqrt{a}} = \sqrt{\frac{-c}{a}}.$$

Solution 2: $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

$$\lim_{b\to 0} \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{0-\sqrt{-4ac}}{2a} = -\frac{\sqrt{-4ac}}{2a} = -\frac{\sqrt{-c}}{\sqrt{a}} = -\sqrt{\frac{-c}{a}}.$$

These are, in fact, the two solutions that result from solving the quadratic equation in which $b = 0$. $\left(ax^2 + c = 0 \Rightarrow x = \pm\sqrt{\frac{-c}{a}}.\right)$ i.e., The "solution function" is continuous at $b = 0$.

Finally, what happens to the two solutions of the quadratic equation if $c \to 0$, while a and b are held constant? Again, we assume that $a > 0$. We consider the case in which $b > 0$. The case for $b < 0$ is similar.

Solution 1: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$$\lim_{c\to 0} \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-b+\sqrt{b^2-0}}{2a} = \frac{-b+\sqrt{b^2}}{2a} = \frac{-b+b}{2a} = 0.$$

Solution 2: $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

$$\lim_{c\to 0} \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b-\sqrt{b^2-0}}{2a} = \frac{-b-\sqrt{b^2}}{2a} = \frac{-b-b}{2a} = -\frac{b}{a}$$

These are, in fact, the two solutions that result from solving the quadratic equation when $c = 0$. $\left(ax^2 + bx = 0 \Rightarrow x\,(ax + b) = 0 \Rightarrow x = 0 \text{ or } x = -\frac{b}{a}.\right)$

If these three limit problems were examined in reverse order, the situation in which $a \to 0$ would be even more impressive. When $b \to 0$ or $c \to 0$, the two quadratic solutions simply approach the two solutions which would result if the linear and constant terms were respectively deleted. A quite different consequence results when $a \to 0$.

This is a good example of the application of the limit concept to a situation familiar to a student before the calculus. The reader and students are invited to find other such examples.

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