Some Simple Proofs of the Sums of Some Alternating Series

By Hussain Elalaoui-Talibi

Introduction

In this paper, we give some simple proofs of the convergence of two classical convergent alternating series with sums $\ln 2$, and $\frac{\pi}{4}$.

The classical series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln 2$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$$

are usually seen in second or third semester Calculus courses, and are approached via power series expansion of $\ln(x+1)$, and $\tan^{-1}(x)$. The proofs of those expansions use Taylor's Remainder Theorem which is pretty advanced. An elementary proof of the sum of these series can be found in [1]. We will give some different elementary proofs of these sums involving geometric series and simple integration.

Two Simple Proofs

THEOREM 1. $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2$.

PROOF. First note that for $k \in \mathbf{N}$, $\int_0^1 x^{k-1} dx = \frac{1}{k}$, so that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} &= \sum_{k=1}^{\infty} \int_0^1 (-1)^{k-1} x^{k-1} dx \\ &= \int_0^1 \left(\sum_{k=1}^{n} (-1)^{k-1} x^{k-1} \right) dx. \end{aligned}$$

Using the formula for the sum of the first n terms of a geometric series, we have

$$\sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1} = \frac{1-(-x)^n}{1-(-x)} = \frac{1}{1+x} - \frac{(-x)^n}{1+x}$$

so that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} &= \int_0^1 \left(\frac{1}{x+1} - \frac{(-x)^n}{1+x} \right) dx \\ &= \ln 2 - \int_0^1 \frac{(-x)^n}{1+x} dx, \qquad (1) \end{aligned}$$

but

$$\left| \int_0^1 \frac{(-x)^n}{1+x} dx \right| \leq \int_0^1 \frac{x^n}{1+x} dx \leq \int_0^1 x^n dx = \frac{1}{n+1}.$$

If we let $n \to \infty$ in eq. 1, the result follows. □

THEOREM 2. $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$.

PROOF. First we observe that $\int_0^1 x^{k-\frac{1}{2}} dx = \frac{1}{k+\frac{1}{2}}$, so that $\frac{1}{2} \int_0^1 x^{k-\frac{1}{2}} dx = \frac{1}{2k+1}$, and it follows that

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} &= \sum_{k=0}^{n} \frac{1}{2} \int_0^1 (-1)^k x^{k-\frac{1}{2}} dx \\ &= \frac{1}{2} \left(\int_0^1 \sum_{k=0}^{n} (-1)^k x^{k-\frac{1}{2}} \right) dx. \end{aligned}$$

Using the formula for the sum of the first n terms of a geometric series, we have

$$\sum_{k=0}^{\infty} (-1)^k x^{k-\frac{1}{2}} = \frac{x^{-1/2}(1-(-x)^{n+1})}{1-(-x)} = \frac{x^{-1/2}}{1+x} - \frac{x^{-1/2}(-x)^{n+1}}{1+x},$$

so that

$$\begin{aligned}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1} &= \frac{1}{2}\left(\int_0^1 \frac{x^{-1/2}}{1+x}dx - \int_0^1 \frac{x^{-1/2}(-x)^{n+1}}{1+x}dx\right)\\ &= \frac{1}{2}\left(\frac{\pi}{2} - \int_0^1 \frac{x^{-1/2}(-x)^{n+1}}{1+x}dx\right)\\ &= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{x^{-1/2}(-x)^{n+1}}{1+x}dx, \qquad (2)\end{aligned}$$

since $\int_0^1 \frac{x^{-1/2}}{1+x}dx = \frac{\pi}{2}$, which can be verified using the substitution $u = x^{1/2}$. Now,

$$\left|\int_0^1 \frac{x^{-1/2}(-x)^{n+1}}{1+x}dx\right| \leq \int_0^1 \frac{x^{n+\frac{1}{2}}}{1+x}dx \leq \int_0^1 x^{n+\frac{1}{2}}dx = \frac{1}{n+\frac{3}{2}}.$$

So let $n \to \infty$ in eq. 2, and the result follows. □

References

[1] Zheng, Liu, An Elementary Proof for Two Basic Alternating Series. *Amer. Math. Monthly* **109** (2002) 187-188.

Department of Mathematics
Jermaine J. Ferguson
Tuskegee University
Tuskegee, AL 36088
talibi@tuskegee.edu
(334)-727-8212