Perfect Matchings - Enumeration and Properties of Extendability

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Abstract

A *perfect matching* of a graph G is an independent set of edges in G that will cover all of the vertices of G. Early studies of perfect matchings centered on the problem of enumeration. How many perfect matchings are contained in a given graph G? While the problem has since been shown to be computationally difficult for general graphs G, it has been studied for infinite families of special graphs. We begin by using a helpful classification method to count the number of perfect matchings in several infinite families of graphs. For graphs of the form $G = H \times K_2$, we use another classification method to again help us count the number of perfect matchings in the graph G. More recently, studies of perfect matchings have included defining characterizations of n-extendable graphs. A graph G of order greater than $2n$ and having a perfect matching is defined to be *n-extendable* if every independent set of edges of size n can be extended to a perfect matching in G. We explore the properties of extendability in certain Cartesian product graphs.

Introduction

We will consider only finite, undirected, connected graphs without loops or multiple edges. For terms or notation not defined here, see [1]. For a graph G, we will denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. When two vertices are connected by an edge they are said to be *adjacent.* Similarly, when two edges share a vertex they are said to be *adjacent.* For adjacent vertices

u and v in the graph G, we will denote the edge between them by uv.

Given a vertex v in a graph G, the *neighborhood* of v, denoted $N(v)$, is the set $\{w \in V(G) : w \text{ is adjacent to } v\}$. A set F of edges of G is said to *cover* $V(G)$ if every vertex of G belongs to at least one edge from F. An *independent* set of edges (or vertices) in G is a set of edges (or vertices) of G which are pairwise non-adjacent.

For a positive integer k, a *k-matching* of G is an independent set of edges of size k. A *perfect matching* (or 1-factor) of a graph G is an independent set of edges in G that will cover all of the vertices of G. Some obvious conditions on G for a perfect matching to exist are that $|V(G)|$ must be even and the degree of each vertex must be at least one. While these conditions are necessary they are certainly not sufficient. For example, the graph in Figure 1 has no perfect matchings.

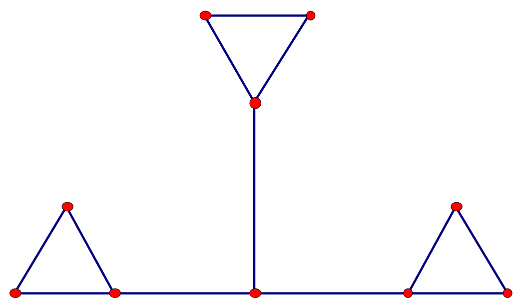


Fig. 1

For the following definitions, $G_1$and G_2 are two graphs with disjoint vertex sets. The Cartesian product $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either

$$u_1 = v_1 \text{ and } u_2v_2 \in E(G_2) \quad \text{or} \quad u_2 = v_2 \text{ and } u_1v_1 \in E(G_1).$$

As in Plummer [2], a graph G of order greater than $2n$ and having a perfect matching is defined to be *n-extendable* if every independent set of edges of size n can be extended to a perfect matching in G. Let $S \subseteq V(G)$. By $G - S$ we will mean the graph with vertex set $V(G) - S$ and edge set $\{uv : uv \in E(G) \text{ and } u \notin S \text{ and } v \notin S\}$.

A Method for Enumerating Perfect Matchings

We use a helpful classification method to count the number of perfect matchings in a graph G. Every vertex v must be covered by a perfect matching, which means there is exactly one vv_i, for some $v_i \in N(v)$, in every perfect matching. We know that the number of perfect matchings containing vv_i equals the number of perfect matchings in $G - \{v, v_i\}$. Therefore, for any $v \in V(G)$, we can classify the perfect matchings of G by which edge vv_i they contain. Throughout the paper we will let $f(G)$ be the number of perfect matchings in the graph G. So, for any fixed vertex v of G,

$$f(G) = \sum_{w \in N(v)} f(G - \{v, w\})$$

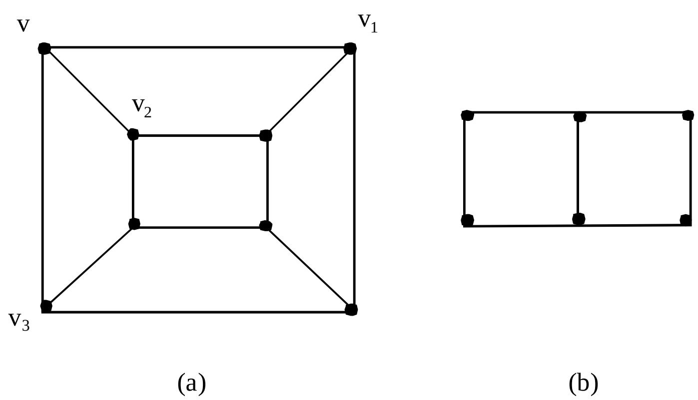


Fig. 2

For example, consider the graph Q_3 (Figure 2a). For any $vw \in E(Q_3)$, the graph $Q_3 - \{v, w\}$ is isomorphic to $P_3 \times K_2$ (Figure 2b). Using the labeling of Figure 2a we see that $f(Q_3) = f(Q_3 - \{v, v_1\}) + f(Q_3 - \{v, v_2\}) + f(Q_3 - \{v, v_3\})$. Thus,

$$f(Q_3) = 3 \cdot f(P_3 \times K_2) = 9.$$

Now consider the graph $K_{n,n}$, the complete bipartite graph, where one part has vertex set $\{v_1, v_2, \ldots, v_n\}$ and the other part has vertex set $\{w_1, w_2, \ldots, w_n\}$. Consider vertex v_1 with $N(v_1) = \{w_1, w_2, \ldots, w_n\}$. Using our method, it follows that $f(K_{n,n}) = \sum_{i=1}^{n} f(K_{n,n} - \{v_1, w_i\})$. Realizing that for every $vw \in E(K_{n,n})$, we have $K_{n,n} - \{v, w\} \cong K_{n-1,n-1}$, we can see that $f(K_{n,n}) = n \cdot f(K_{n-1,n-1})$. When we expand further we get $f(K_{n,n}) = n(n-1)(n-2)\ldots(2) f(K_{1,1})$. The graph $K_{1,1}$ has one perfect matching, so

$$f(K_{n,n}) = n!.$$

Similarly, for K_n (n-even), $f(K_n) = (n-1) f(K_{n-2})$. Expanded, $f(K_n) = (n-1)(n-3)\ldots(3) f(K_2)$. The graph K_2 has one perfect matching, so

$$f(K_n) = (n-1)(n-3)\ldots(3)(1) \qquad (n\text{-even}).$$

We also use this method on several classes of Cartesian product graphs. One example is the graph $P_3 \times P_n$. For this example we will define T_m as $(P_3 \times P_{m+1}) - \{v\}$, where $v \in V(P_3 \times P_{m+1})$ has degree two. (See Figure 3a.) Before we calculate $f(P_3 \times P_n)$ we will calculate $f(T_m)$.

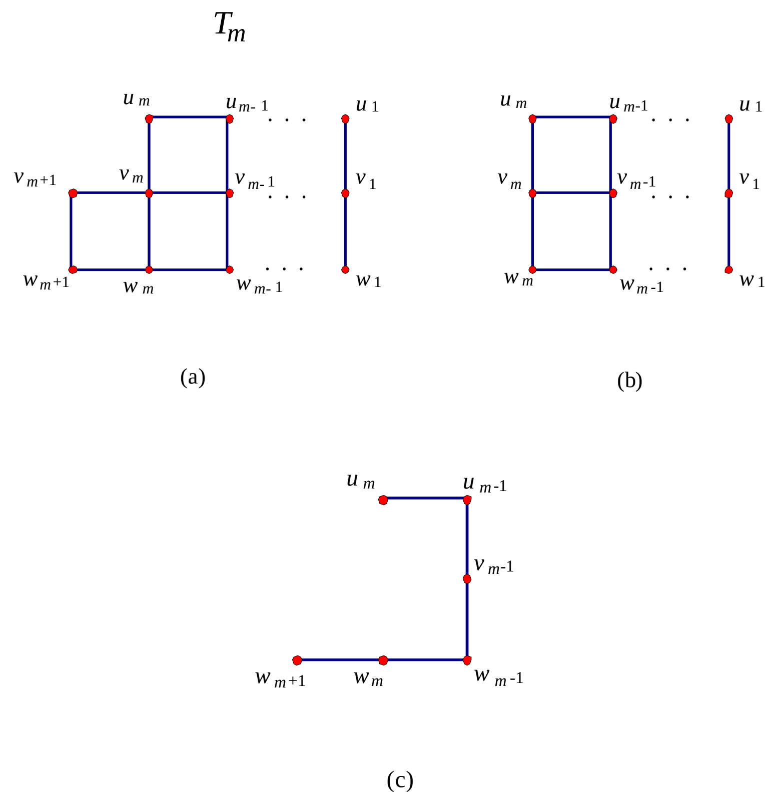
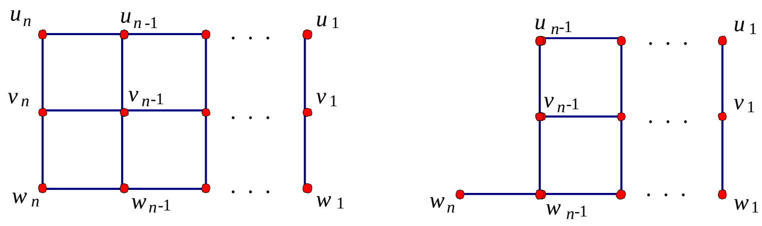


Fig. 3

With reference to Figure 3, observe that: $f(T_m) = f(T_m - \{v_{m+1}, w_{m+1}\}) + f(T_m - \{v_{m+1}, v_m\})$, and $f(T_m - \{v_{m+1}, w_{m+1}\}) = f(P_3 \times P_m)$. (See Figure 3b.) Next observe that $f(T_m - \{v_{m+1}, v_m\}) = f(T_{m-2})$ because, as we see in Figure 3c, each perfect matching of $T_m - \{v_{m+1}, v_m\}$ must contain the edges $u_m u_{m-1}$ and $w_{m+1} w_m$. But, finally, we note that the graph $(T_m - \{v_{m+1}, v_m\}) - \{u_m, u_{m-1}, w_{m+1}, w_m\}$ is isomorphic to T_{m-2}. Therefore,

$$f(T_m) = f(P_3 \times P_m) + f(T_{m-2}).$$

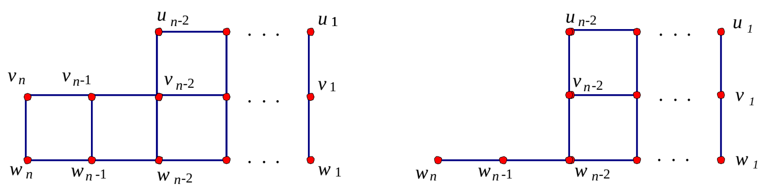


(a) (b)

Fig. 4

Now we will calculate $f(P_3 \times P_n)$. (See Figure 4a.) If $P_3 \times P_n$ is to have a perfect matching, n must be even. Then $f(P_3 \times P_n) = f(P_3 \times P_n - \{u_n, v_n\}) + f(P_3 \times P_n - \{u_n, u_{n-1}\})$ .We will look at each of the terms of the sum individually.

First we will consider $f(P_3 \times P_n - \{u_n, v_n\})$. (See Figure 4b.) We can see that $f(P_3 \times P_n - \{u_n, v_n\}) = f(T_{n-2})$ because the edge $w_n w_{n-1}$ must be a part of any perfect matching and $(P_3 \times P_n - \{u_n, v_n\}) - \{w_n, w_{n-1}\}$ is isomorphic to T_{n-2}.



(a) (b)

Fig. 5

Now consider $f(P_3 \times P_n - \{u_n, u_{n-1}\})$, which must be broken down further. Let $R = (P_3 \times P_n - \{u_n, u_{n-1}\})$. (See Figure 5a.) Then $f(R) = f(R - \{v_n, w_n\}) + f(R - \{v_n, v_{n-1}\})$. We see that $f(R - \{v_n, w_n\}) = f(T_{n-2})$ since $R - \{v_n, w_n\}$ and T_{n-2} are isomorphic. Now we note that $f(R - \{v_n, v_{n-1}\}) = f(P_3 \times P_{n-2})$, because, as we can see in Figure 5b, any perfect matching of $R - \{v_n, v_{n-1}\}$ must include the edge $w_n w_{n-1}$ and $(R - \{v_n, v_{n-1}\}) - \{w_n, w_{n-1}\}$ is isomorphic to $P_3 \times P_{n-2}$.

Combining these results we see that

$$f(P_3 \times P_n) = f(T_{n-2}) + f(T_{n-2}) + f(P_3 \times P_{n-2})$$

or that

$$f(P_3 \times P_n) = 2f(T_{n-2}) + f(P_3 \times P_{n-2}). \tag{1}$$

Recall from our prior work that

$$f(T_m) = f(P_3 \times P_m) + f(T_{m-2}). \tag{2}$$

From eq. 2 we can derive that $f(T_{n-2}) = f(P_3 \times P_{n-2}) + f(T_{n-4})$. Substituting this result into eq. 1 we conclude that

(3)

$$\begin{aligned} f(P_3 \times P_n) &= 2\left(f(P_3 \times P_{n-2}) + f(T_{n-4})\right) + f(P_3 \times P_{n-2}) \\ &= 3f(P_3 \times P_{n-2}) + 2f(T_{n-4}). \end{aligned}$$

From eq. 1 we can derive that $2f(T_{n-4}) = f(P_3 \times P_{n-2}) - f(P_3 \times P_{n-4})$. Making a final substitution of this result into eq. 3 we get our result:

$$f(P_3 \times P_n) = 4f(P_3 \times P_{n-2}) - f(P_3 \times P_{n-4}).$$

Using similar strategies we were able to discover recurrence relations for $f(P_2 \times P_n)$, $f(P_2 \times C_n)$, and $f(P_4 \times P_n)$.

The following is a summary of the results using this method of classification.

	n-even	n-odd
$f(C_n)$	2	0
$f(K_n)$	$(n-1)(n-3)\ldots(3)(1)$	0
$f(K_{n,n})$	$n!$	$n!$
$f(P_2 \times P_n)$	$f(P_2 \times P_{n-1}) + f(P_2 \times P_{n-2})$	same
$f(P_3 \times P_n)$	$4f(P_3 \times P_{n-2}) - f(P_3 \times P_{n-4})$	0
$f(P_4 \times P_n)$	$f(P_4 \times P_{n-1}) + 5f(P_4 \times P_{n-2}) + f(P_4 \times P_{n-3}) - f(P_4 \times P_{n-4})$	same
$f(P_2 \times C_n)$	$f(P_2 \times C_{n-1}) + f(P_2 \times C_{n-2})$	$f(P_2 \times C_{n-1}) + f(P_2 \times C_{n-2}) - 2$

Perfect Matchings in Graphs of the Form $G = H \times K_2$

For graphs of the form $G = H \times K_2$, we use another classification method to help in counting the number of perfect matchings. In $G = H \times K_2$ there are two copies of the graph H which we will call H_v and H_w. Let $V(H_v) = \{v_1, v_2, \ldots, v_n\}$ and let $V(H_w) = \{w_1, w_2, \ldots, w_n\}$, where v_i and w_i and are corresponding vertices in H. (i.e. $v_iv_j \in E(H_v)$ if and only if $w_iw_j \in E(H_w)$.) Also, the edge $v_iw_j \in E(H \times K_2)$ if and only if $i = j$. We will call all edges not completely contained in H_v or H_w *crossover* edges. (e.g., v_1w_1 is a crossover.)

For this type of graph we classify the perfect matchings by the number of edges used from H_v and H_w. Note that when $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and a perfect matching M contains exactly k edges from H_v, then M contains exactly k edges from H_w. So, for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, let F_k be the set of all perfect matchings of G each of which contains exactly k edges from H_v. Then we get the following result:

$$f\left(G\right) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left| F_k \right|.$$

For example, if $G = P_4 \times K_2$ (See Figure 6), then

$$\begin{array}{ll} F_0 = \{\{v_1w_1, v_2w_2, v_3w_3, v_4w_4\}\} & |F_0| = 1 \\ F_0 = \{\{v_1v_2, w_1w_2, v_3w_3, v_4w_4\}, \{v_1w_1, v_2v_3, w_2w_3, v_4w_4\}, & |F_1| = 3 \\ \quad \{v_1w_1, v_2w_2, v_3v_4, w_3w_4\}\} & \\ F_2 = \{\{v_1v_2, v_3v_4, w_1w_2, w_3w_4\}\} & |F_2| = 1 \end{array}$$

It follows that $f\left(P_4 \times K_2\right) = 1 + 3 + 1 = 5$

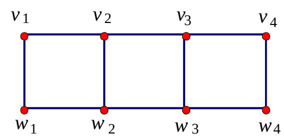


Fig. 6

Now consider the general graph $P_n \times K_2$, where P_n is the path $v_1, v_2, v_3, \ldots, v_n$. In order to compute $f\left(G\right)$ using the above method, we need to determine the size of F_k for all k, $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. We let $h_{k,n}$ be the number of k-matchings of P_n. Then $h_{1,n} = n-1$, the number of edges in P_n.

Note that every k-matching either includes v_1v_2 or it does not. The number of k-matchings of P_n that include v_1v_2 equals the number of $(k-1)$-matchings of $P_n - \{v_1, v_2\}$. That number is $h_{k-1,n-2}$. The number of k-matchings of P_n which do not include the edge v_1v_2 equals the number of k-matchings of $P_n - \{v_1\}$. That number is $h_{k,n-1}$. Therefore, $h_{k,n} = h_{k-1,n-2} + h_{k,n-1}$. Using an induction argument we can show that $h_{k,n} = \binom{n-k}{k}$ for $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

In $P_n \times K_2$, $\quad |F_k| = h_{k,n}$ since any perfect matching of $P_n \times K_2$ which includes exactly k edges from H_v must also include $n - 2k$ crossover edges. The $2k$ vertices of h_w left uncovered by these crossover edges can be covered by a k-matching of H_w in exactly one way, which is to use the edges corresponding to those used in H_v.

So,

$$f\left(P_n \times K_2\right) = \textstyle\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} |F_k| = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} h_{k,n} \ \text{(where } h_{0,n} = 1,\ h_{1,n} = n-1\text{)}$$

$$= \textstyle\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k}.$$

For $G = C_n \times K_2$, let $g_{k,n}$ be the number of k-matchings of C_n. Again, in this case, we see that $|F_k| = g_{k,n}$. So $f(C_n \times K_2) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g_{k,n}$ where $g_{0,n} = 1$; $g_{1,n} = n$; and $g_{k,n} = g_{k,n-1} + g_{k-1,n-2}$. It can also be shown that $g_{k,n} = 2h_{k,n} - h_{k,n-1} = 2\binom{n-k}{k} - \binom{n-k-1}{k}$. Consequently,

$$f(C_n \times K_2) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[2\binom{n-k}{k} - \binom{n-k-1}{k} \right].$$

For $G = K_{n,n} \times K_2$, let $b_{k,n}$ be the number of k-matchings of $K_{n,n}$. Clearly $b_{1,n} = n^2$, the number of edges in $K_{n,n}$. With a similar argument to that used for $P_n \times K_2$, we can show that for k such that $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$,

$$b_{k,n} = \frac{\left(\frac{n!}{(n-k)!} \right)^2}{k!}.$$

Now any perfect matching of $K_{n,n} \times K_2$ which includes exactly k edges from H_v must also include $n - 2k$ crossover edges. When the vertices of H_w that are covered by these crossover edges are removed from H_w, the graph that remains is $K_{k,k}$. From an earlier result we know that these $2k$ vertices can be covered $f(K_{k,k}) = k!$ ways. So, $|F_k| = b_{k,n} \cdot k!$. Combining these results we see that

$$f(K_{n,n} \times K_2) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\left(\frac{n!}{(n-k)!} \right)^2}{k!} \cdot k! = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{n!}{(n-k)!} \right)^2.$$

Extending Perfect Matchings

We will now change our focus from counting perfect matchings to properties of extendability in certain Cartesian product graphs. We will make use of the following theorem of Plummer [2].

THEOREM 1. *If $n \geq 2$ and G is n-extendable, then G is also $(n-1)$-extendable.*

We begin by characterizing the values of r for which the graph $K_n \times K_2$ is r-extendable.

THEOREM 2. *The graph $K_n \times K_2$ is $\lfloor \frac{n}{2} \rfloor$-extendable.*

PROOF. We use the same notation as before, letting $G = H \times K_2$, where H_v and H_w are copies of K_n. Let M be any set of $\lfloor \frac{n}{2} \rfloor$ edges in $K_n \times K_2$. The graph $K_n \times K_2$ is $\lfloor \frac{n}{2} \rfloor$-extendable if any such M extends to a perfect matching.

Let $|M| = x + y + z$, where x is the number of edges from H_v in M, y is the number of crossover edges in M, and z is the number of edges from H_w in M.

Case 1 *The set M does not contain crossover edges* $(y = 0)$.

Subcase 1 *Suppose n is even.*
If x (or z) $= \lfloor \frac{n}{2} \rfloor$, then M is a perfect matching of H_v (or H_w). To extend this to a perfect matching of $K_n \times K_2$, add edges that are a perfect matching of H_v (or H_w). If $x \neq \lfloor \frac{n}{2} \rfloor$ and $z \neq \lfloor \frac{n}{2} \rfloor$, then $x < \lfloor \frac{n}{2} \rfloor$ and $z < \lfloor \frac{n}{2} \rfloor$. Because K_n is clearly $\left(\lfloor \frac{n}{2} \rfloor - 1\right)$-extendable, it follows from Theorem 1 that K_n is x-extendable and z-extendable. Thus, M can be extended to a perfect matching of $K_n \times K_2$ by extending the set of x edges in H_v and the set of z edges in H_w.

Subcase 2 *Suppose n is odd.*
Since $2 \cdot \lfloor \frac{n}{2} \rfloor < n$, there exists j, $1 \leq j \leq n$, such that neither v_j nor w_j is incident to any edge in M. To extend M to a perfect matching of $K_n \times K_2$, add the crossover $v_j w_j$ and then extend in the graphs $K_{n-1} = H_v - \{v_j\}$ and $K_{n-1} = H_w - \{w_j\}$ separately.

Case 2 *The set M contains at least one crossover edge* $(y \neq 0)$.
Let $v_1w_1, v_2w_2, \ldots, v_yw_y$ represent the crossover edges in M. We can extend M to a perfect matching if we can extend the set of edges from M in the graph $K_n \times K_2 - \{v_1, v_2, \ldots, v_y, w_1, w_2, \ldots, w_y\}$ to a perfect matching. Call this set of edges M'. The remaining graph is of the form $K_{n-y} \times K_2$. Since there are no crossovers from the remaining graph in the set M' and since we deleted only crossover edges from our original graph, we know that M' has $x + z = \lfloor \frac{n}{2} \rfloor - y$ edges. This means that $x \leq \lfloor \frac{n}{2} \rfloor - y$ and $z \leq \lfloor \frac{n}{2} \rfloor - y$ where x is the number of edges from M in the original $H_v = K_n$ and from our present $H_{v^*} = K_{n-y}$, and z is the number of edges from M in the original $H_w = K_n$ and from our present $H_{w^*} = K_{n-y}$.

We showed in Case 1 that for M, an $(x+y+z)$-matching of $K_n \times K_2$, if $y = 0$ and $x + z = \lfloor \frac{n}{2} \rfloor$, then M can be extended to a perfect matching of $K_n \times K_2$. Thus for the current graph $K_{n-y} \times K_2$ and the $(x+z)$-matching M', if we can show that $x + z \leq \lfloor \frac{n-y}{2} \rfloor$, then by Theorem 1 and Case 1 we can extend M' to a perfect matching of $K_{n-y} \times K_2$, from which it then follows that M can be extended to a perfect matching of G when

$y \neq 0$. It is easy to show that since $x + z = \left\lfloor \frac{n}{2} \right\rfloor - y$ and $y \geq 1$, it must be the case that $x + z \leq \left\lfloor \frac{n-y}{2} \right\rfloor$. We have now shown that $K_n \times K_2$ is $\left\lfloor \frac{n}{2} \right\rfloor$-extendable.

□

THEOREM 3. *If $r > \left\lfloor \frac{n}{2} \right\rfloor$, then $K_n \times K_2$ is not r-extendable.*

PROOF. To show that $K_n \times K_2$ is not r-extendable for any $r > \left\lfloor \frac{n}{2} \right\rfloor$, it follows from Theorem 1 that we need only to show that $K_n \times K_2$ is not $\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$-extendable.

If n is odd let $M = \{v_1v_2, v_3v_4, \ldots, v_{n-2}v_{n-1}, w_1w_n\}$, and if n is even let $M = \{v_1v_2, v_3v_4, \ldots, v_{n-3}v_{n-2}, v_{n-1}w_{n-1}, w_{n-2}w_n\}$. In each case M is an $\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$-matching of $K_n \times K_2$ which covers all of the neighbors v_n of but does not cover v_n. Therefore M cannot be extended to a perfect matching. □

An equivalent statement to Theorem 3 is that if $K_n \times K_2$ is r-extendable, then $r \leq \left\lfloor \frac{n+2}{2} \right\rfloor - 1$. For our final result we give a generalization of this theorem.

THEOREM 4. *If $K_n \times K_m$ is r-extendable, then $r \leq \left\lfloor \frac{n+m}{2} \right\rfloor - 1$.*

PROOF. We will show that $K_n \times K_m$ is not $\left\lfloor \frac{n+m}{2} \right\rfloor$-extendable. We will assume that m is even and that m is at least 4. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_m) = \{w_1, w_2, \ldots, w_n\}$. Then $V(K_n \times K_m) = \{(v_i, w_j) : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. To make the notation easier to handle, we let $x = (v_1, w_1)$, $a_i = (v_i, w_1)$, and $b_j = (v_i, w_j)$ for $2 \leq i \leq n$ and $2 \leq j \leq m$.

Case 1 *The integer n is even.*

Let M be the following set of independent edges:

$$M = \{a_2a_3, a_4a_5, \ldots, a_{n-2}a_{n-1}, b_2b_3, b_4b_5, \ldots, b_{m-2}b_{m-1}, a_nz, b_my\},$$

where $z = (v_n, w_2)$ and $y = (v_2, w_m)$. Then $|M| = \left\lfloor \frac{n+m}{2} \right\rfloor$ and all of the neighbors of x are covered by M. It is clear that M cannot be extended to a perfect matching of $K_n \times K_m$.

Case 2 *The integer n is odd.*

Here we let M be the set of independent edges given by

$$M = \{a_2a_3, a_4a_5, \ldots, a_{n-1}a_n, b_2b_3, b_4b_5, \ldots, b_{m-2}b_{m-1}, b_my\},$$

where $y = (v_2, w_m)$. As in the previous case $|M| = \left\lfloor \frac{n+m}{2} \right\rfloor$, and all of the neighbors of x are covered by M and so M is not contained in any perfect matchings of$K_n \times K_m$.

These cases together show that $K_n \times K_m$ is not $\left\lfloor \frac{n+m}{2} \right\rfloor$-extendable and thus is not r-extendable for any $r > \left\lfloor \frac{n+m}{2} \right\rfloor$. $\square$

References

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