No Repetitions. I Repeat: No Repetitions.

By Mike Hudak

Introduction

Whilst a fledgling graduate student oh-so-many years ago, a conversation with friend and physicist Dr. Philip Honsinger regarding the game of chess prompted the following question: does there exist an infinite sequence of finitely many objects containing no consecutive repetitions of any finite length?

With only one or even two objects available, the answer is rather trivially 'no' since, for example, the finite sequence 0,1,0 of the two objects '0' and '1' cannot be extended without either '0' or the block '0,1' consecutively repeating. Clearly sequences like 0,0 and 1,1 already violate the repetition condition and the case involving the sequence 1,0,1 is similar to the case involving 0,1,0 with the roles of '0' and '1' reversed.

But what if more objects are available? Say, for example, the ten decimal digits 0 through 9. One might naturally gravitate towards the irrational numbers, given their common description of having "non-repeating, non-terminating decimal expansions." But, lo and behold, such familiar irrationals as π, e, and roots of prime numbers have finite consecutive repetitions. π has consecutive 3's in the twenty-fourth and twenty-fifth decimal positions, $e = 2.718281828\ldots$, where one can already notice the block '1,8,2,8' consecutively repeating, $\sqrt{13} = 3.60555\ldots$ with '5's' clearly repeating. Evidently, an alternate approach is needed.

Whereas we had ardently hoped that the question was open, alas, to our chagrin, the question had been answered in the affirmative years earlier. Nevertheless, we feel that our solution at the time, presented here for the four object case, provides an instructive example of a non-trivial result established via the application of elementary techniques only, although the construction is somewhat ingenious. The students just familiarizing themselves with the rudiments of proof theory and set theory will be exposed to an infinite union; the notion of ordinals as sets; and definition by recursion; as well as proofs by induction, by multiple cases, by contraposition, and by contradiction.

Preliminaries

We adhere to standard set-theoretic notions and notations, with some minor departures (which we clearly specify) for convenience only.

ω is the set of finite ordinals, commonly referred to by students as the set of whole numbers. Thus, $\omega = \{0, 1, 2, 3, ...\}$, endowed with the familiar well-ordering.

Each finite ordinal (whole number) is the set of smaller finite ordinals. Thus: $0 = \{\} = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, $4 = \{0, 1, 2, 3\}$, and so on. Note that a whole number expressed in exponential form is also the set of smaller whole numbers. For example, $2^3 = 8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

As always, a *function* is a set of ordered pairs, with no two pairs having the same first coordinate but (perhaps) different second coordinates. A *sequence* is a function with domain ω. A finite sequence is a function with domain a finite ordinal. Thus, a finite sequence with domain n is a (finite) sequence of length n. Indexing begins with 0, not 1, for sequences and finite sequences in this paper.

For sets A, B, ${}^{A}B = \{f : A \to B\}$. In particular, ${}^{8}4 = \{f : 8 \to 4\} = \{f : \{0, 1, 2, 3, 4, 5, 6, 7\} \to \{0, 1, 2, 3\}\} =$ {all finite sequences of length 8 (indexing beginning with 0) of the objects 0,1,2,3}. Note that the size of the *set* ${}^{8}4$ is the *number* 4^8.

We refer to objects in consecutive positions within a (finite) sequence as a *block* and, if the block consecutively repeats, the second occurrence is called the *twin block.*

The Construction

We define an *identification function* $f : 4 \to^8 4$ as follows:

$$\begin{aligned}
f(0) &= \langle 0, 1, 2, 1, 0, 2, 1, 2 \rangle = 01210212 \\
f(1) &= \langle 0, 1, 2, 3, 0, 2, 1, 3 \rangle = 01230213 \\
f(2) &= \langle 0, 1, 2, 1, 0, 2, 3, 1 \rangle = 01210231 \\
f(3) &= \langle 0, 1, 3, 2, 0, 2, 1, 3 \rangle = 01320213,
\end{aligned}$$

where the ordering symbols '<' and '>' have been deleted for convenience and the commas have been dispensed with as well. No ambiguity shall arise from this as the only objects in the sequences we construct are the single digits 0,1,2,3. Note, for example, that $f(0)(0) = 0$, $f(0)(1) = 1$, $f(0)(7) = 2$, $f(1)(5) = 2$, etc. We may employ subscripts for clarity. Thus, $f_0(0) = 0$, $f_0(1) = 1$, etc. (Keep in mind that our indexing begins with 0, hence, $f_0(0) = 0$ is the 0^{th} term of $f(0)$, $f_0(5) = 2$ is the 5^{th} term of $f(0)$, and so on.) Clearly, f identifies, or associates, a particular sequence of length 8 of the objects 0,1,2,3 with each of the ordinals 0,1,2,3.

Next, we use the above function f to recursively define a sequence of finite sequences, where, for each $n = \omega$, $s_n : 8^n \to 4$.

$s_0 = \langle 0 \rangle = 0$, where, again employing an abuse of notation for convenience, we will drop the ordering symbols, but be aware that s_0 is not the empty set, but rather is a sequence of length 1 with $s_0(0) = 0$. Note that $8^0 = 1 = \{0\}$ and $0 \in 4 = \{0,1,2,3\}$ so s_0 is a function from 8^0 to 4 as desired.

Given $n \in \omega$ and $s_n : 8^n \to 4$, define $s_{n+1} : 8^{n+1} \to 4$ as follows: Fix $j \in 8^{n+1} = \left\{0, 1, 2, \ldots, 8^{n+1} - 1\right\}$. By the division algorithm, there exist unique integers, say q and r, with $q \in 8^n = \{0, 1, 2, \ldots, 8^n - 1\}$ and $r \in 8 = \{0, 1, 2, ..., 7\}$, such that $j = 8q+r$. Define $s_{n+1}(j) = [f[s_n(q)]](r) = f_{s_n(q)}(r)$.

Note: s_n is a finite sequence with domain 8^n and codomain 4. Since $q \in 8^n$, $s_n(q) \in 4$. So $f[s_n(q)]$ is a sequence with domain 8 and codomain 4 (namely, one of the four sequences listed at the beginning of this section). Since $r \in 8$, $[f[s_n(q)]](r) \in 4$ as desired.

That was the official, technical definition of the sequence $\langle s_n, n \in \omega \rangle$ for those readers wishing to construct precise proofs of subsequent results. For instance, it can be verified by induction that each s_{n+1} extends s_n as a function and that the pattern 0,1,_,_,0,2,_,_ recurs in blocks of length 8 throughout each s_{n+1}. We leave the details to the reader.

Although officially defined above, for clarity and ease of communication of general ideas, we content ourselves in this paper with the following conceptual, intuitive notion of the construction of $\langle s_n, n \in \omega \rangle$. In general, s_{n+1} is built from s_n by successively inputting the digits of the sequence s_n into the identification function f and concatenating the resulting sequences. For shorthand, we denote this process as $s_{n+1} = f(s_n)$, where

$$\begin{aligned}
s_0 &= 0 \ (= \langle 0 \rangle, \text{ technically) to start with, and} \\
s_1 &= f(s_0) = f(0) = 01210212, \\
s_2 &= f(s_1) = f(01210212) \\
&= f(0)\, f(1)\, f(2)\, f(1)\, f(0)\, f(2)\, f(1)\, f(2) \\
&= 01210212, 01230213, 01210231, 01230213, 01210212, \\
&\quad\ 01210231, 01230213, 01210231,
\end{aligned}$$

where the commas were placed after every 8 positions for emphasis on understanding the construction process. So s_2 is a sequence of length $8^2 = 64$ and $s_3 = f(s_2)$ is a sequence of length 8^3, since each of the 64 digits in s_2 is replaced by a sequence of length 8 and then concatenated. It is imperative that the reader understand this construction algorithm in order to follow the ensuing reasoning processes.

We now use $\langle s_n, n \in \omega \rangle$ to construct an infinite sequence, $g : \omega \to \{0,1,2,3\}$. Whereas an analyst might think to define g as the limit of the sequence of finite functions under some type of functional convergence, a set theorist uses the union operation instead. Thus, $g = \cup_{n\in\omega} s_n$. Note that each s_n is a function with domain ${}^n8$ and as such, is technically a set of ordered pairs. For instance, $s_0 = \{(0,0)\}$ and $s_1 = \{(0,0),(1,1),(2,2),(3,1),(4,0),(5,2),(6,1),(7,2)\}$, whence $s_0 \cup s_1 = s_1$. This ensures that g is a relation and that the domain of g so defined is ω. The fact that each s_{n+1} extends s_n ensures that g is a function, not merely a relation. Also note that g extends each s_n by virtue of its construction as the union of the s_n's.

The Main Result

We now focus our attention on the establishment of the following theorem:

THEOREM 1. *For each $n \in \omega$, s_n contains no consecutive repetition of any finite length.*

The proof is by induction with several cases needing to be considered, and we sketch the reasoning involved, relying heavily on the heuristic description of the construction of the sequence $\langle s_n, n \in \omega \rangle$ rather than the rather cumbersome, but technically more accurate, official definition.

By inspection, $s_0 = 0$ (again, technically $\langle 0 \rangle$) has no repetitions so the basis step is trivial. The inductive step requires the verification of the statement: *For each $n \in \omega$, if s_n contains no consecutive repetition of any finite length, then s_{n+1} also contains no consecutive repetition of any finite length.*

We choose to instead prove the equivalent contrapositive formulation: *For each $n \in \omega$, if s_{n+1} contains a consecutive repetition of some finite length, then s_n also contains a consecutive repetition of some finite length.*

Fix $n \in \omega$. Assume that s_{n+1} contains a consecutive repetition of some finite length (referred to as a block). We must show that s_n contains a consecutive repetition of some finite length as well. We focus our attention on the block within s_{n+1} assumed to consecutively repeat. There are multiple cases to consider, namely, that the repeating block begins with a:

- 0
- Successor to a 0
- Double successor to a 0
- Triple successor to a 0 (i.e., a predecessor to a 0).

Suppose the block begins with 0. As no 0's consecutively repeat in any of the $f(j)$'s, for $j \in \{0,1,2,3\}$, and as each $f(j)$ begins with 0 but ends in a number other than 0, concatenating the $f(j)$'s in the construction of s_{n+1} from s_n cannot induce a consecutive repetition of the number 0. Thus, the block in s_{n+1} must be longer than length 1. As the number 0 only occurs in s_{n+1} in every fourth position, and as its twin (i.e., the consecutive repetition of the block) must also begin with 0 in the case under consideration, we argue that the block must be at least length 4. However, since 0's appear in s_{n+1} only in the pattern 0,1,_,_,0,2,_,_, and the twin is an exact duplicate of the block, it is clear that the block must be at least length 8 and, indeed, be of length a multiple of 8, say $8k$, for some$k \in \omega - \{0\}$, in order for the 0's and successors to the 0's in the block to align with their respective occurrences in the twin.

There are two subcases to consider, namely, the block begins with:

- 01
- 02.

Assume the block begins with 01. In view of the construction of s_{n+1} using f-values of consecutive digits in s_n and concatenating, there must exist a block of digits $i_0, i_1, i_2, \ldots, i_{k-1}$ in s_n for which $f(i_0)\, f(i_1)\, f(i_2) \ldots f(i_{k-1})$ constitutes the repeating block in s_{n+1}. Since f is one to one, the twin in s_{n+1} must also be formed as $f(i_0)\, f(i_1)\, f(i_2) \ldots f(i_{k-1})$, in which case the $i_0, i_1, i_2, \ldots, i_{k-1}$ must consecutively repeat in s_n, as desired.

Next, assume the repeating block in s_{n+1} begins with 02. As the last four positions of each of $f(0)$, $f(1)$, $f(2)$, $f(3)$, are distinct patterns of 0,2,_,_, we infer that the first four digits of the block are in actuality the final four digits of exactly one of $f(0)$, $f(1)$, $f(2)$, or $f(3)$. Thus, we infer the existence of a consecutive sequence of digits in s_n, namely, $i_0, i_1, \ldots, i_{k-1}, i_k$, for which the block in s_{n+1} begins with the final four digits of $f(i_0)$, has middle portion $f(i_1)\, f(i_2) \ldots f(i_{k-1})$, and terminates with the first four digits of $f(i_k)$. Since the twin begins with the final four digits of $f(i_k)$, and the twin must begin in an identical manner as the block, we infer the digit i_k is the same digit as i_0. Thus the block terminates with the first four digits of $f(i_0)$. Then, essentially shifting to the left four positions from where the original block begins, we assert that $f(i_0) f(i_1) f(i_2) \ldots f(i_{k-1})$ constitutes a new block in s_{n+1} beginning with a 01 formation and which also consecutively repeats. This is the subcase already completed.

Comic Relief

The manner in which the second subcase was handled by appeal to the first subcase reminds us of the following anecdotal tale.

A physicist and a mathematician were to be analyzed regarding their respective problem solving strategies. A series of tests was devised. Initially, a pail of water was situated on a tabletop. On the other side of the room was a stove. The problem was to transfer the pail of water from the tabletop to the stove. The physicist simply picked up the pail, awkwardly carried it across the room, and placed it on the stove. Problem solved. The mathematician did likewise. Little information was gleaned from this.

Next, the pail was placed at the foot of the table. Again, the objective was to transfer the pail across the room to the stove. The physicist bent over, raised the pail, and toted it across the room to the stove, as before. The pail was then returned to its original position at the base of the table. The mathematician, however, when confronted with the same situation, walked to the table, raised the pail and placed it on the tabletop and stopped. After some time, the experimenters, fearing a lack of communication in the objective, suggested that the pail now needed to be transferred from the tabletop to the stove. In response, the mathematician replied, "Yes, but *that* problem has already been solved!"

Case by Case

Continuing onward to the remaining cases, we trust that the relevance of the above will become apparent. As a reminder, in all of the following cases, the repeating block must have length a multiple of 8, say $8k$, to ensure that the 0's align properly in the adjacent twin.

We note also that some shifting of starting positions of blocks will take place in the arguments. Consider the example

_,_, 1,2,3,4,1,2,3,4,_,_ with the block 1234 of length 4 consecutively repeating. If the starting position of the old block is shifted two positions to the left, to verify that the new block, still of length 4, also repeats, it is only necessary to check the alignment of the two newly acquired positions on the left, all previous alignments remaining intact. Thus, for the block _,_,1,2 to consecutively repeat with 3,4,1,2 in the above example, the initial two blanks must be filled with 3,4 but the previous alignment of the remaining positions need not be re-checked.

Consider now the case where the repeating block in s_{n+1} begins with a successor to a 0 — i.e., the position directly to the left of the start of the block is occupied by a 0. Recalling that 0's only occur every fourth position, to ensure proper alignment of the twin with the original block, the twin must also begin with the successor of a 0. Thus a shift of the original block one unit to the left will result in a new block of the same length (but beginning with 0) that also consecutively repeats. Thus we are in the case where we have a repeating block in s_{n+1} that begins with 0, and as in the anecdote, that case has already been solved.

Suppose the repeating block instead begins with a double successor to a 0 — i.e., the position two places to the left of the start of the block is occupied by a 0. Again, it is clear that the twin must also begin with the double successor to a 0.

First note that the two positions following 0,1,_,_ in each of the four f-sequences are occupied by 21, 23, 21, and 32, respectively, whereas those following the 0,2,_,_ are occupied by 12, 13, 31, and 13, respectively. As these two lists are disjoint, we infer that the start of the block and the start of its twin must both be preceded by the same pair, be it 01 or 02. In either instance, shifting the start of the block two units to the left (keeping the same length) creates a new repeating block with initial 0, and that is a completed case.

Lastly, suppose the block begins with the triple successor to a 0. We rely heavily on inspection of the actual four f-sequences in all future cases and subcases. We analyze three cases, namely, that the block begins with a 1, a 2, or a 3, respectively. Each case will be subdivided into two subcases, namely, where the 2^{nd} and 3^{rd} positions of the block are 01 or 02, respectively.

In case 1, subcase 1, we argue that the initial 101 of the block and its twin must both be preceded by 023, by inspection of the f-sequences (only f_2 ends in 1). Thus a shift to the left 3 units creates a new repeating block with initial 0, and we have already solved that case!

In case 1, subcase 2, we argue that the initial 102 of the block and its twin must both be preceded by 012 (whether from f_0 or from f_2 is immaterial). Again a shift to the left 3 units creates a new repeating block with initial 0, a solved case.

In case 2, subcase 1, the initial 201 of the block and its twin must both be preceded by 021 (only f_0 ends in 2). Shifting 3 units to the left again yields a repeating block with initial 0.

In case 2, subcase 2, the initial 202 of the block and its twin must be from f_3 and hence preceded by 013, so shifting left 3 units yields a new repeating block with initial 0.

In case 3, subcase 1, the initial 301 of the block and its twin must be preceded by 021 (from f_1 or from f_3), whence, shifting 3 units left again produces a repeating block with initial 0.

In case 3, subcase 2, the initial 302 of the block and its twin must be 012 from f_1 and once more a shift to the left of 3 units results in a new repeating block with initial 0.

All cases closed! (Whew!)

The Final Analysis

We have now verified that each of the *finite* sequences, s_n, contains no repeating block of any finite length. However, we promised an *infinite* sequence of four objects with no repeating block of any finite length. We assert that $g = \cup_{n\in\omega} s_n$ fits the bill. The proof is by contradiction. For suppose that g has a block of some finite length with a repeating twin. Since g extends each s_n, each s_n has domain 8^n, and the limit of the sequence of 8^n's as $n \to \infty$ is ∞; there must exist a positive integer, say j, which is sufficiently large so that the repeating block and its twin in the g sequence is also contained in the s_j sequence. But this contradicts the main result that none of the s_n's contains a repeating block. Therefore g can have no consecutively repeating block of any finite length, as claimed.

In conclusion, we remark that the reader may have noticed that the situation where only two objects were available to construct the sequence was quite easily dispensed with in the negative (it was not possible to construct a sequence with the desired properties) and the situation with four objects available was answered in the affirmative (hence also for *more* than four as well). Clearly, only the situation with exactly three objects available remains. We invite readers to investigate for themselves, but we note that the result is known.

Department of Mathematics
207 Hodge
University of South Carolina Upstate
Spartanburg, SC
MHUDAK@uscupstate.edu