Right Triangles With Inradius r

By Stephen B. Curry

Introduction

The origins of this paper are in a problem often given to students: to show that the incircle of a 3 - 4 - 5 triangle has radius 1. In this paper we combine geometry and number theory to find all the right triangles with integer length sides whose incircle has a given radius r, where r is an integer. In addition, we count the number of primitive right triangles having incircle with radius r.

Pythagorean Triples

A *Pythagorean triple* is a set of three positive integers $\{a, b, c\}$ such that $a^2 + b^2 = c^2$. A Pythagorean triple is called *primitive* if, and only if, a, b, and c are pairwise relatively prime. That is, if none of the pairs $\{a, b\}$, $\{b, c\}$, or $\{a, c\}$ share a positive factor greater than 1.

It is easy to see that if $\{a, b, c\}$ is a Pythagorean triple, and if any two of $\{a, b, c\}$ have a common factor k, then k is a factor of the third, as well. Thus, in order to show that a Pythagorean triple $\{a, b, c\}$ is primitive, it is sufficient to show that a single pair from $\{a, b, c\}$ is relatively prime.

Suppose $\{a, b, c\}$ is a Pythagorean triple. If a and b are both odd, then a^2+b^2 is even, but not divisible by 4, and so not a perfect square (since all perfect squares are either of the form $4n$ or $4n+1$). Thus, a and b can not both be odd. If both a and b are even, then c must also be even, and we have the following well-known result:

Proposition 1. *If $\{a, b, c\}$ is a primitive Pythagorean triple, then one of a and b is even, the other is odd, and c is odd.*

Throughout, we shall adopt the convention that in a primitive Pythagorean triple $\{a, b, c\}$, a is even, and b is odd.

The following well-known result can be found in most basic number theory texts.

THEOREM 1. *Suppose a, b, and c are integers, with a even and b odd. In order that $\{a, b, c\}$ be a primitive Pythagorean triple, it is both necessary and sufficient that there exist positive integers m and n so that:*

(1) m and n are relatively prime,
(2) $m \not\equiv n \bmod 2$
(3) $a = 2mn$; $b = m^2 - n^2$; $c = m^2 + n^2$.

If, for example, we select $m = 2$ and $n = 1$, we get the familiar triple $\{4, 3, 5\}$; with $m = 3$ and $n = 2$, we get $\{12, 5, 13\}$. We say that m and n *generate* the triple $\{a, b, c\}$ and that m and n generate the triangle whose sides have lengths a, b, and c. It is clear that distinct choices of m and n generate distinct triples.

Positive integers m and n that do not satisfy conditions 1 and 2 of the theorem, but do satisfy condition 3, generate a triple that is not primitive. For example, $m = 5$ and $n = 3$ yields $\{30, 16, 34\}$; $m = 6$ and $n = 3$ yields $\{36, 27, 45\}$. It is interesting to note that not all non-primitive triples can be generated by values of m and n. For example, there do not exist m and n that generate the triple $\{12, 9, 15\}$. For, if $2mn = 12$, then m and n must be either 6 and 1 or 3 and 2. 6 and 1 generate $\{12, 35, 37\}$, while 3 and 2 generate $\{12, 5, 13\}$.

Geometry and Triples

The *incircle* of a right triangle T is the circle inscribed in T. The center of the incircle of a triangle T is called the *incenter* of T, and the radius of the incircle of T is called the *inradius* of T.

We shall say that a right triangle whose sides form a primitive Pythagorean triple is a *primitive triangle.*

In what follows, we are interested only in triangles whose sides are of integer length.

The following property of the inradius of a triangle is a widely used exercise.

PROPOSITION 2. *Suppose T is a right triangle with sides that form a Pythagorean triple $\{a, b, c\}$ (that is, sides of integer length). The inradius r of T is an integer.*

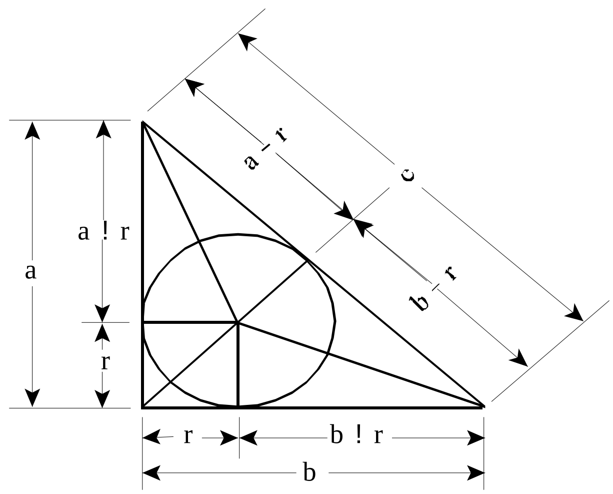

Fig. 1

PROOF. Congruent triangles give us that $c = a + b - 2r$, so that $r = \frac{1}{2}(a + b - c)$. If a, b, and c are all even, so that the triple is not primitive, then $a+b-c$ is even and positive. If a is even and b is odd, so that c is odd, then $a+b$ is odd, so that, again, $a+b-c$ is even and positive. Thus, r is an integer. $\square$

Number Theory Meets Geometry

Direct substitution from the proposition gives us the following.

LEMMA 1. *Suppose T is a primitive triangle whose sides are generated by m and n, and suppose T has inradius r. Then $r = n(m-n)$.*

We note that since $m \not\equiv n \bmod 2$, $(m-n)$ is odd. Thus, we have the following.

LEMMA 2. *Suppose T is a primitive triangle generated by m and n, and suppose T has inradius r. Then r is even if, and only if, n is even.*

Suppose $r = 2^k \cdot q$, where q is the product of powers of odd primes. We know that n has a factor of 2. If $n = 2^t \cdot q_1$, where $t < k$ and q_1 is a factor of q, then $m - n = 2^{k-t} \cdot q_2$, where $q_1 q_2 = q$ and $k - t \geq 1$. However, since $m - n$ must be odd, this is impossible, and we have shown the following.

LEMMA 3. *Suppose r is an even positive integer, and T is a primitive triangle so that T has inradius r, and T is generated by m and n. If 2^k divides r, then 2^k divides n.*

Taking $r = 2^k$, we immediately have the following.

THEOREM 2. *Suppose $r = 2^k$. There is exactly one primitive triangle T with inradius r, and T is generated by $n = 2^k$ and $m = 2^k + 1$.*

PROOF. Suppose T has inradius r, and T is generated by m and n. By Lemma 3 we know that 2^k divides n. Since $n(m-n) = r$ with $m - n$ a positive integer, $n \leq r$. Thus, $n = 2^k$, and $m - n = 1$. □

For example, consider $r = 4$. We get that $n = 4$, and $m = 5$. These values generate the primitive triple $\{40, 9, 41\}$, and the primitive triangle with sides of length 40, 9, and 41 is the only primitive triangle with inradius 4.

We now consider the odd prime factors of r.

LEMMA 4. *Suppose r is a positive integer with prime factorization $r = 2^k \cdot p_1^{q_1} \cdot p_2^{q_2} \cdot \ldots \cdot p_t^{q_t}$, and T is a primitive triangle so that T has inradius r, and T is generated by m and n. Then if p_i divides n, $p_i^{q_i}$ divides n.*

PROOF. For convenience we write $r = 2^k \cdot p_i^{s_1} \cdot p_i^{s_2} \cdot p_j \cdot p_o$, where $s_1 + s_2 = q_i$, and $s_1 \geq 1$. The other odd prime factors of r are represented as $p_j \cdot p_o$.

Suppose that n has $p_i^{s_1}$ as a factor, but has no higher power of p_i as a factor. Since $n(m-n) = r$, and $2^k \mid n$, we must have

$$n = 2^k \cdot p_i^{s_1} \cdot p_j \text{ and } n(m-n) = p_i^{s_2} \cdot p_o$$

where one of p_j and p_o could equal 1. This gives

$$m = 2^k \cdot p_i^{s_1} \cdot p_j + p_i^{s_2} \cdot p_o$$

so that s_2 must equal zero, else m and n have p_i as a common divisor (and the triple is not primitive). Thus, $s_1 = q_i$, and $p_i^{q_i}$ divides n. □

We note that it is not necessary for any factor of $r = 2^k \cdot p_1^{q_1} \cdot p_2^{q_2} \cdot \ldots \cdot p_t^{q_t}$ to be a factor of n, except 2^k, and we may select any of the $p_i^{q_i}$ to be factors of n, and we will generate a primitive triangle whose inradius is r. Furthermore, different selections of $p_i^{q_i}$ as factors of n yield different values of m and n, generating different primitive triangles, each of which has inradius r. Hence, to count the number of primitive triangles having inradius r, we need only count the number of ways we can select factors of n. We have the following.

THEOREM 3. *Suppose r is a positive integer with prime factorization $r = 2^k \cdot p_1^{q_1} \cdot p_2^{q_2} \cdot \ldots \cdot p_t^{q_t}$. There are exactly 2^t primitive triangles with inradius r.*

PROOF. Each subset of $\{1, 2, \ldots, t\}$ (of which there are 2^t) defines a unique collection of the $p_i^{q_i}$, and each of these collections yields a unique value of n. For each n, we find a unique m, since $n(m-n) = r$; each pair m, n generates a unique primitive triple, giving a primitive triangle with inradius r. Furthermore, we have seen that for a primitive triangle T to have inradius r, it must be generated in this fashion. □

We quickly have the following corollary.

COROLLARY 1. *If p is an odd prime, there are exactly two primitive triangles with inradius p.*

We shall see that there is a third right triangle with integer-length sides — though not primitive — with inradius p, and for values of r that are not odd primes there are other right triangles with integer-length sides and inradius r, but first we look at an example.

EXAMPLE 1. *Let us take $r = 90 = 2 \cdot 3^2 \cdot 5$. There are two odd primes in the prime factorization of 90, so we will find $4 = 2^2$ primitive triangles with inradius 90. The generator n must have 2 as a factor, and it can have factors $3^2, 5$, both of these, or neither of them.*

n	$m-n$	m	a; b; c
2	45	47	188; 2205; 2213
$2 \cdot 3^2$	5	23	828; 205; 853
$2 \cdot 5$	9	19	380; 261; 461
$2 \cdot 3^2 \cdot 5$	1	91	16,380; 181; 16,381

We have counted exactly how many primitive triangles have a given inradius r, and we have shown how to find all of these triangles by selecting values for n using the prime factorization of r. We shall now investigate non-primitive right triangles and their inradii. Our goal is to find all the right triangles whose sides are of integer length and whose inradius is some given integer r.

We know that for a given Pythagorean triple $\{a, b, c\}$, the triangle with sides of length a, b, and c has inradius given by $r = \frac{1}{2}(a + b - c)$. We see from this relation that for a positive integer k, the triangle with sides of length ka, kb, and kc has inradius kr. For example, a $\{4, 3, 5\}$ triangle has inradius 1, so a $\{4r, 3r, 5r\}$ triangle has inradius r.

To find all the right triangles with sides of integer length that have inradius r, we look again to the prime factorization of r and find all the positive factors of r, including 1. For each factor t of r, we find all the primitive triangles with inradius t. We then multiply the lengths of the sides of these triangles by $\frac{r}{t}$, and we claim that this set of triangles is the complete set of triangles whose sides are of integer length and whose inradius is r. We first see that the algorithm does give us triangles with inradius r.

Suppose t is a factor of r so that $r = k \cdot t$, and T is a primitive triangle with sides of length a, b, and c, such that T has inradius t. Then the triangle kT with sides of length ka, kb, and kc has inradius r. Thus, each of the triangles obtained by this method does, indeed, have the desired inradius r.

On the other hand, if T is a triangle with inradius r, then either T is primitive, or it is not. If it is primitive, we have found it with our algorithm, since r is a factor of itself. If T (with sides of length a, b, and c) is not primitive, the lengths of the sides have a greatest common divisor, $k > 1$. The triangle T' with sides of length $a' = \frac{a}{k}$, $b' = \frac{b}{k}$, and $c' = \frac{c}{k}$ is a primitive triangle with inradius $\frac{r}{k}$. We know that $\frac{r}{k}$ must be an integer, and so a factor of r. Thus, we would find the triangles T' and then T with our algorithm.

Hence, we have found the complete set of triangles whose sides are of integer length and whose inradius is r. This suggests the following.

Proposition 3. *If r is an odd prime, there are exactly three right triangles with sides of integer length having inradius r; there are exactly two right triangles with sides of integer length having inradius 2.*

Proof. There are only two positive factors of r : 1 and r. There are two primitive triangles with inradius r, and there is one primitive triangle with inradius 1. □

We can generalize this proposition to include powers of primes.

Theorem 4. *Suppose p is an odd prime, and k is a positive integer. There are exactly $2k+1$ right triangles with sides of integer length that have inradius p^k; there are exactly $k+1$ right triangles with sides of integer length that have inradius 2^k.*

Proof. The factors of p^k are $l, p, p^2, \ldots, p^k$. There is one primitive triangle with inradius 1, and for $i \geq 1$, there are two primitive triangles with inradius p^i. The factors of 2^k are $1, 2, 2^2, \ldots, 2^k$. For each $i \geq 0$, there is one primitive triangle with inradius 2^i. □

We demonstrate the algorithm for finding all the right triangles with inradius $r = 90$.

EXAMPLE 2. *Let* $r = 90 = 2 \cdot 3^2 \cdot 5$. *We know that* $2^k \cdot p_1^{q_1} \cdot p_2^{q_2} \cdot \ldots \cdot p_t^{q_t}$ *has* $(k+1) \cdot \prod_{i=1}^{t} (q_i + 1)$ *factors, so we expect* $(1+1)(2+1)(1+1) = 12$ *factors. The set of factors of* 90 *is* $\{1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90\}$.

$x =$ *factor of* 90	*factorization of* x	# *primitives with inradius* r
1	1	1
2	2	1
3	3	2
5	5	2
6	$2 \cdot 3$	2
9	3^2	2
10	$2 \cdot 5$	2
15	$3 \cdot 5$	4
18	$2 \cdot 3^2$	2
30	$2 \cdot 3 \cdot 5$	4
45	$3^2 \cdot 5$	4
90	$2 \cdot 3^2 \cdot 5$	4
	Total	*30*

We have already found the four primitives associated with 90; the other 26 triangles are non-primitive. For example, for $x = 1$, we get $\{4, 3, 5\}$, so the associated triangle with inradius 90 has sides $\{360, 270, 450\}$. For $x = 2$, we find the primitive $\{12, 5, 13\}$; the associated triangle with inradius 90 has sides $\{540, 225, 585\}$. For $x = 3$, we find two primitive triples: $\{8, 15, 17\}$ and $\{24, 7, 25\}$. The associated triangles with inradius 90 have sides $\{240, 450, 510\}$ and $\{720, 210, 750\}$, respectively. We will not burden the reader with the other 22 triangles with inradius 90.

Further Considerations

We have described all the right triangles with sides of integer length whose inradius is a given integer r, and it is reasonable to investigate triangles whose sides have rational length. However, it is not a fruitful endeavor. For each positive integer r, there are infinitely many right triangles with sides of rational length whose inradius is r.

For let p be a positive integer, and let T be a primitive triangle with sides $\{a, b, c\}$ and inradius p. The triangle T' with sides $\left\{\frac{a}{p}, \frac{b}{p}, \frac{c}{p}\right\}$ has inradius 1, so the triangle T'' with sides $\left\{\frac{ra}{p}, \frac{rb}{p}, \frac{rc}{p}\right\}$ has sides of rational length and has inradius r.

Department of Mathematics
Alabama School of Fine Arts
1800 8th Ave. North
Birmingham, AL 35203-2203
(205)252-9241