Results as By-Products

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The main objective of the current paper is to present a series of interesting mathematical results which have unusual origins and, in particular, to consider those results which came out as by-products from another (sometimes very remote) area of Mathematics. The nature of this paper is expository. Hence, most proofs are omitted. The motivation for results discussed herein, originates from observations made in the course of our collective scientific experience, and from facts from the history of mathematics, not generally known by mathematicians.

Often a complicated work contains results from another branch of mathematics. Sometimes these results will have direct proofs in their own field. Other times no direct proof of this result is known. We will present examples of both kinds of results. This phenomenon may have both objective and subjective (psychological) explanations. It is not uncommon for a result in one area of mathematics to include a result from another area of mathematics that comes as an unexpected corollary. For example, the Fourier series for $\sin^2 x$ provides, "at no extra cost," a proof that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Sometimes a natural proof of a by-product result can be obtained later using the tools and terminology of the result's native field. We will refer to this as the *ordinary case.* Other times we have the *extraordinary case,* when only a proof from another field is available at the moment. It is known that the famous Desargues plane theorem [**11**] can be proven by the stereometric approach, but a proof by pure geometric planar methods has not yet been found. Thus, this result has remained extraordinary for a long time.

The great Pierre de Fermat wrote the following passage (see [**18**]) in the margin of a page in his copy of the book *Arithmetica* by Diophantos: "To divide a cube into two other cubes, a fourth power, or, in general, any power whatever into two powers of the same denomination above the second is impossible, and I surely have found an admirable proof of this, but the margin is too narrow to contain it."

We contemplate the possibility that this famous margin note may have referred to an extraordinary result from a quite different consideration. At least this is more palatable than the possibility that the great mathematician was mistaken.

We will present examples of both ordinary and extraordinary results. In addition, we will present an extraordinary result that we hope will have an ordinary future.

At the end of the 1960's, R. James conjectured that for any normalized basis in a Hilbert space H, the basic harmonic series converges. Recall, a basis in a Banach space X is a sequence $\{e_k\}_{k=1}^{\infty}$ such that each $x \in X$ has a unique decomposition $x = \sum_k \alpha_k e_k$. R. James's conjecture means that for any normalized basis $\{e_k\}_{k=1}^{\infty}$, ($||e_k|| = 1$ for all k), the series

$$\sum_{k=1}^{\infty} \frac{e_k}{k}$$

converges. This result is obvious if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis. One simply applies Parseval's identity. For an arbitrary normalized basis, this result was much more difficult to prove.

The proof came about, not by directly studying series in Hilbert spaces, but as a by-product result from trying to generalize Parseval's identity for uniformly convex Banach spaces. A Banach space X is said to be *uniformly convex* if there exists a function $\delta(w) > 0$ on $(0, 2]$ such that for any normalized $x, y \in X$ with $||x - y|| \geq w$ we have

$$1 - \frac{||x + y||}{2} \geq \delta(w).$$

Examples of uniformly convex Banach spaces are L_p and l_p where $1 < p < \infty$. The following theorem was obtained in [**7**] and a little later independently in [**12**], and allows us to answer R. James' conjecture quite easily.

Theorem 1. *If X is a uniformly convex Banach space and $\{e_k\}_{k=1}^{\infty}$ is a normalized basis in X, then there exists $1 < r \leq s < \infty$ and $0 < A, B < \infty$ such that if $x = \sum_k \alpha_k e_k$ then*

$$B\left(\sum_k |\alpha_k|^s\right)^{\frac{1}{s}} \leq ||x|| \leq A\left(\sum_k |\alpha_k|^r\right)^{\frac{1}{r}} \tag{0.1}$$

A basis $\{e_k\}_{k=1}^{\infty}$ in Hilbert space H is called *equivalent* to an orthonormal or Riesz basis if inequality (0.1) holds for $r = s = 2$. By the inequality (0.1) one can conclude that the above mentioned harmonic series in Hilbert space H converges. In addition, we can see that other series in H such as

$$\sum_{k=1}^{\infty} \frac{e_k}{k^\alpha}, \alpha > \alpha_0 \text{ for some } \alpha_0 \in (0,1)$$

and

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} e_k$$

also converge. In each case we do not know an elementary direct proof and therefore we would say this is still an extraordinary result. We think that, in explaining this "by-product phenomenon," it is essential to consider also a subjective (psychological) reason. The following humorous anecdote illustrates this.

In 1927, J. Schauder [9] constructed the well known basis in $C[0,1]$ of all continuous functions on $[0,1]$ with the max-norm. During the same period, the following conjecture, known as "Schauder's conjecture," was formulated: *For any compact metric space K, the space $C(K)$ of continuous functions on K with the max-norm has a basis.* In attempting to solve this problem, many important "side" results were obtained. But in 1952, independently of this problem, A. Miliutin established (published in 1966) the following important result:

Theorem 2. (A. Miliutin, [7]) *For a compact metric space K with the cardinality of the continuum, the space $C(K)$ is isomorphic to $C[0,1]$. In other words, there exists an algebraic isomorphism (a bijection preserving vector operations) $T : C(K)$ onto $C[0,1]$ such that $||T|| < \infty$ and $||T^{-1}|| < \infty$.*

Obviously, from this brilliant theorem follows the solution of Schauder's conjecture for the case of a compact metric space K with cardinality of the continuum, but A. Miliutin didn't know about that problem as he was only focused on the isomorphism problems.

In 1966, Miliutin considered the following "by-product" problem: *For which compact metric space K does $C(K)$ have an interpolating basis $\{e_k(t)\}_1^\infty$ with given sequence of nodes $\{t_k\}_1^\infty \subset K$ (i.e. in the decomposition of each $x(t) \in C(K)$*

$$x(t) = \sum_1^\infty \alpha_k e_k$$

the sum $\zeta = \sum_{k=1}^n \alpha_k e_k$ interpolates $x(t)$ at nodes $t_1, t_2, \ldots t_n$ with $n = 1, 2, \ldots$)? He got the following answer:

THEOREM 3. *$C(K)$ has an interpolating basis with nodes $\{t_k\}_1^\infty$ if and only if $\{t_k\}_1^\infty$ is dense in K.*

Of course this theorem completely solves the J. Schauder problem, but the author overlooked this and was informed about it only because of a reference on his publication [**17**] in an article by E. Michael and A. Pelczynski [**16**] which appeared at almost the same time. By the way, in that article [**16**], they proved the following big strengthening of the J.Schauder conjecture (also as a side product of one important geometric discovery):

THEOREM 4. *$C(K)$ has a monotone basis $\{e_k\}_1^\infty$ (i.e. for any sequence of scalars $\{\alpha_k\}_1^\infty$), the sequence*

$$\{||\sum_{k=1}^n \alpha_k e_k||\}_{n=1}^\infty$$

is non-decreasing.

A space $C(K)$ is a particular case of the important concept of Banach space which is, by definition, a complete normed linear space. In 1931, Stephen Banach formulated his famous "basis problem" on the existence of a basis in every separable Banach space.

For the next 40 years, many efforts were made to solve this problem using the tools from the geometry of Banach spaces. Using this natural approach, mathematicians were only successful in finding a negative solution for the case of the monotone basis problem [**9**]. In 1972, a negative solution of the "basis problem" was suddenly obtained by Per Enflo as a by-product result of his negative solution of Grothendieck's famous "boundary approximation problem" in operator theory. For details we refer readers to the original work [**10**]. For his solution of the basis problem, Per Enflo got a special symbolic award from Polish mathematicians, the "fried Mazur goose," which according to tradition, should be eaten alone or shared with colleagues. Since the goose was large, Per opted for the latter. Some of his colleagues joked that, for the Grothendieck problem, Per was eligible for a larger award — such as fried ostrich (we think that this by-product of the Warsaw Zoo was not available). But the following conclusion is not a joke: sometimes a by-product result has the form of a Corollary of a more general fact from quite a different area.

Independent of such curious phenomena, the provocative role of by-product results even from close fields of mathematics sometimes looks very impressive.

There is an interesting story that illustrates the connection between different areas of Mathematics as a source of by-product results. In 1937, Professor Mark Krein taught a special course on Operator Theory at Odessa State University (Odessa, Ukraine). At that time, the spectral analysis of self-adjoint operators acting on Hilbert spaces had just been developed, but nothing was known about non-self-adjoint operators. Mark Krein told his students that the person had not yet been born who would say something about the infinite-dimensional analogy of Jordan theory. He might have been right in his statement if Moshe Livsic had not been among his students in this course. In 1946, Livsic defended his dissertation to get a Doctor of Science Degree in Mathematics, the highest degree in Mathematics in the former USSR. The defense was in Moscow at the Steklov Mathematical Institute and his referees were Stephen Banach, Israel Gelfand and Abraham Plesner. In his dissertation, M. Livsic made the first fundamental steps in the theory of non-self-adjoint operators, introducing the major unitary invariant of a non-self-adjoint operator-characteristic function. Later, in 1954, he published his classic paper [**13**] on spectral decomposition of bounded non-self-adjoint operators which was the starting point for the infinite-dimensional version of Jordan theory. In this paper, he established an infinite-dimensional analogue of the classical I. Schur theorem that any matrix is unitary equivalent to a triangular one (matrices A and B are unitary equivalent if there is a unitary matrix U such that $A{=}U^{-1}BU$). As a by-product result of M. Livsic's triangular model came his remarkable criterion for completeness of eigenvectors and associated vectors for a wide class of bounded, linear operators. Later, a direct proof (without the triangular model and the characteristic matrix-valued functions machinery) was obtained by one of his Ph.D. students making this situation an ordinary one. The triangular model was used to find, for the first time, conditions under which eigenvectors and associated vectors of a bounded linear operator form a Riesz basis. Later, direct proofs were obtained in a more general setting by various mathematicians [**5**], [**6**], [**14**], [**15**].

Another remarkable result that came out of the M. Livsic triangular model approach was his theorem that any completely non-self-adjoint volterra operator (compact operator with only zero points of spectrum) with one dimensional imaginary part is unitary equivalent to the operator of integration $(Jf)(x) = i\int_x^l f(t)dt$ in $L_2[0,l]$. This operator, as was proven by M. Brodskii [**1**], is the infinite-dimensional analogue of the operator with one Jordan block (all nontrivial invariant subspaces of this operator form a chain).

Trying to describe all invariant subspaces of the operator J by the method of characteristic functions, M. Brodskii [**2**] unexpectedly obtained, as a by-product result, the solution of I. Gelfand's problem: Find criterion on a given function in $L_2[0,1]$ for which $(J^n f)(x)$, $(n = 0, 1, 2, ...)$ is dense in $L_2[0,1]$.

This example confirms the conjecture that by-products are one of the essential factors which make the prognosis of developments in mathematics somewhat unreliable.

In [**20**] and [**21**], the following inequality was established and came free as a by-product result of the theory of non-self-adjoint contractive extensions of Hermitian contractions and their characteristic functions that have been applied to the above mentioned operator of integration J. For any $l \in (0, \frac{\pi}{2})$ and $f(t) \in L_2[0,l]$

$$\cot l \left| \int_0^l f(t)dt \right|^2 \le \int_0^l |f(t)|^2 \, dt - \int_0^l \left| \int_x^l f(t)dt \right|^2 dx \tag{0.2}$$

The inequality (0.2) is *sharp*, in the sense that the constant, $\cot l$, cannot be improved for any $l \in (0, \frac{\pi}{2})$ and $f(t) \in L_2[0,l]$.

In addition, although [**21**] establishes the inequality, it does not provide necessary and sufficient conditions under which equality holds. Recently, we found a direct approach, which provides an elementary proof of (0.2) and shows that equality holds if and only if $y = C\cos(t - l)$, where C is an arbitrary constant. So, we can consider this as an ordinary result. The following sharp inequality appeared in [**4**]. For all $f(x)$, such that $f''(x) \in L_2[0,\infty)$ and $f(0) = 0$,

$$|f'(0)| \le 2^{-\frac{1}{4}} \left(\int_0^\infty |f(x)|^2 dx + \int_0^\infty |f''(x)|^2 dx \right)^{\frac{1}{2}} \tag{0.3}$$

This inequality was obtained as a by-product result of the extension theory of positive symmetric operators acting on a Hilbert space and spectral analysis of differential operators. No direct proof of (0.3) is known to date, nor is the function known for which (0.3) becomes an equation. We hope that a proof is found soon, but so far this is an extraordinary situation.

We are sure that many mathematicians have some collection of by-product results and that new information on this matter would be very interesting and useful to the mathematical community.

In conclusion, we would like to make the following remarks:

(1) Results as by-products are not a rare phenomenon in mathematics and we think that almost every mathematician has in his (her) scientific biography such a situation.
(2) After getting an essential result, the importance and sensibleness of taking a look at possible applications in quite different areas of mathematics cannot be overestimated. We should ask ourselves: "Is it possible that I have randomly made a discovery or solved some unsolved problem in another field?"
(3) It would be useful to create some system of information about successful by-product results, maybe in the form of corresponding sections in some mathematical journals.

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