Alabama Journal of Mathematics
Activities

The Integration of Graph Theory Into Secondary Mathematics Courses

By Sue Bessler & Tony Thompson

1. Introduction

Graph Theory is an area of mathematics utilizing the use of networks or vertex-edge graphs to display relationships. These relationships can range from social to mechanical in nature. The vertex-edge graphs are a useful, yet under-utilized, representation tool for promoting mathematical reasoning skills. The most popular use of network representations comes in the following forms:

Handshaking Problem - Graphical representation of relationships
Konigsberg Bridge - Eulerian Graphs
Travelers Dodecahedron - Hamiltonian Graphs
Utilities Problem - Planar Graphs
Scheduling Problems - Graph coloring

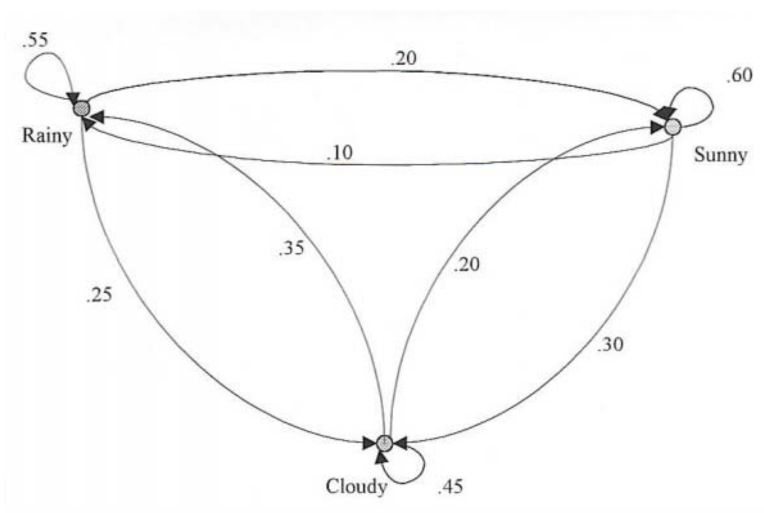
Other interesting connections include the use of matrices to represent relationships. The interconnectedness of matrices, probability, limits, and systems of equations is visible and approachable, through the use of Markov Chains. The following activity provides a relevant, student-friendly, introduction to a Markov Chain application through weather pattern analysis.

Goal: Using information regarding the probability of the occurrence of various weather phenomena tomorrow, based on today's weather conditions, construct a model which gives the probability of the occurrence of various weather phenomena several days into the future, based on today's weather conditions.

Materials: Graphing Calculators (capable of performing matrix operations)

Procedure:

In trailing weather patterns and collecting data on weather phenomena, probabilities of transition from one state to a finite number of other states emerge. For the sake of readability, we will consider three phenomena occurring: cloudy days, C, rainy days, R, and sunny days, S. The representative digraph (directed graph) provides transition probabilities, day to day, from one weather state to another.



Each of the transition probabilities listed, correspond to an edge from one type of weather to another. For example, the edge from cloudy to rainy (C,R), labeled .35, can be interpreted as "35% of cloudy days are followed by a rainy day." The the edge from rainy to sunny (R,S), labeled .20, means that 20% of rainy days are followed by a sunny day.

Suppose today is rainy. What will the weather "most likely" be two days from now?

From the graphical representation we compile the probabilities into the matrix W, with the "from" states forming the rows and the "to" states forming the columns. (i.e., the entry in the i^{th} row and the j^{th} column gives the probability of transition *FROM* the weather state in the i^{th} row *TO* the weather state in the j^{th} column.)

$$\begin{array}{ccc} & & \text{To} \\ & & \begin{array}{ccc} C & S & R \end{array} \\ \text{From} & \begin{array}{c} C \\ S \\ R \end{array} & \begin{bmatrix} .45 & .20 & .35 \\ .30 & .60 & .10 \\ .25 & .20 & .55 \end{bmatrix} \end{array} = W$$

Note that each entry represents the probability of a transition from one state to another. Consequently, the entries are non-negative. Also note that since the probability of the occurrence of each weather pattern appears exactly once in each row, the sum of the entries in each row equals 1.0. A matrix having these characteristics is referred to as a *stochastic matrix.*

As we attempt to predict what will happen two days after a Rainy day, we can follow the directed edges of the digraph and account for all possible compound events. For example, the event of Rainy → Cloudy → Sunny can be extracted from the graph as Rainy → Cloudy = .25, times the probability of the event from Cloudy → Sunny = .20. The probability of that "path" occurring is .05. There do, however, exist other paths that begin with a Rainy day and conclude with a Sunny day two days later. Each one of these need to be accounted for before predictions can be made.

Through the use of matrix multiplication, all such possibilities can be accounted for.

$$\begin{aligned} W^2 = W \cdot W &= \begin{bmatrix} .45 & .20 & .35 \\ .30 & .60 & .10 \\ .25 & .20 & .55 \end{bmatrix} \begin{bmatrix} .45 & .20 & .35 \\ .30 & .60 & .10 \\ .25 & .20 & .55 \end{bmatrix} \\ &= \begin{bmatrix} .35 & .28 & .37 \\ .34 & .44 & .22 \\ .31 & .28 & .41 \end{bmatrix} \end{aligned}$$

The interpretation of the matrix W^2 is shown below:

$$\begin{array}{ccc} & & \text{To} \\ & & \text{(Two Days} \\ & & \text{From Now)} \\ & & \begin{array}{ccc} C & S & R \end{array} \\ \begin{array}{c} \text{From} \\ \text{(Today)} \end{array} & \begin{array}{c} C \\ S \\ R \end{array} & \begin{bmatrix} .35 & .28 & .37 \\ .34 & .44 & .22 \\ .31 & .28 & .41 \end{bmatrix} \end{array} = W^2$$

Note that the entries in each row of W^2 still add up to 1.0 and that W^2 is also a stochastic matrix. The entries in the third row represent a transition from a Rainy day (today) to one of three weather states (indicated by their respective column headings) two days from now. For example, the entry $W_{32} = .28$ implies that if today is Rainy, then there is a 28% probability of a Sunny day occurring two days from now.

This process can be repeated to determine any likelihood k days from the present. The matrix W^k gives the probabilities of the occurrence of each of the three weather states, k days in the future, given any of the three weather states occurring today.

We illustrate this concept with a problem. Given that today is Monday and that it is Cloudy, what will the weather most likely be on Thursday, based on this model? To answer this question, we use the fact that we are interested in the weather three days from now. So we must compute the matrix W^3.

$$W^3 = W \cdot W \cdot W = \begin{bmatrix} .334 & .312 & .354 \\ .340 & .376 & .284 \\ .326 & .312 & .362 \end{bmatrix}$$

The interpretation of the matrix W^3 is shown below:

$$\begin{array}{ccc} & & \begin{array}{c}\text{To} \\ \text{(Three Days} \\ \text{From Now)} \\ \begin{array}{ccc} C & S & R \end{array}\end{array} \\ \begin{array}{c}\text{From} \\ \text{(Today)}\end{array} & \begin{array}{c} C \\ S \\ R \end{array} & \begin{bmatrix} .334 & .312 & .354 \\ .340 & .376 & .284 \\ .326 & .312 & .362 \end{bmatrix} = W^3 \end{array}$$

Reading row 1, representing an initial state of Cloudy, we see that there is a 33.4% chance that it will be Cloudy on Thursday, a 31.2% chance that it will be Sunny on Thursday, and a 35.4% chance that it will be Rainy on Thursday.

An interesting phenomenon occurs as we project further into the future, applying the initial probabilities to indicate transitions. The rows of the matrix W^{30}, for example, are almost identical.

$$W^{30} = \begin{bmatrix} .\bar{3} & .\bar{3} & .\bar{3} \\ .\bar{3} & .\bar{3} & .\bar{3} \\ .\bar{3} & .\bar{3} & .\bar{3} \end{bmatrix}$$

This is indicative of a stabilization occurring over the long-term and a convergence of probabilities of states occurring, in our case, weather on a particular day. This stabilization can be seen in many physical phenomena.

Although weather is dependent on many factors, this model is based solely on using the first state as a indicator of a transition state. That is, we've made the assumption that proceeding from one weather state to the next is dependent only on the current state. This characteristic, along with the existence of a finite number of states, represent a *Markov Chain.*

Enrichment Problem:

Below is a matrix representing the transition probabilities to or from various weather states. Rainy = R, Cloudy = C, Sunny = S, Snowy = N.

$$\begin{array}{cc} & \begin{array}{cccc} R & C & S & N \end{array} \\ \begin{array}{c} R \\ C \\ S \\ N \end{array} & \begin{bmatrix} .05 & .60 & .20 & .15 \\ .50 & .30 & .15 & .05 \\ .10 & .30 & .50 & .10 \\ .30 & .50 & .15 & .05 \end{bmatrix} \end{array}$$

- Produce the associated digraph to represent this situation.
- Find W^2
- Find W^3
- What does the entry W_{12} represent in W^2?
- Today it is Sunny, what will the weather most likely be 2 days from now?
- Today it is Snowing, what will the weather most likely be 3 days from now?
- If the weather is only dependent on the weather the day before, what will the weather most likely be at the end of the month?

References

(1) J.M. Aldous, & R.J. Wilson, Graphs and Applications., Springer-Verlag, London, England, 2000.