Solutions and Discussions

Problem 1 — Volume 26, No. 2, Fall, 2002

Find the volume of a tetrahedron with corners at $(0, 1, 2)$, $(3, 5, 7)$, $(4, 6, 9)$, *and* $(8, 10, 11)$.

Solution

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A tetrahedron can be treated as a pyramid with a triangular base and volume equal to one third the area of the base times the height $\left(V = \frac{1}{3}Bh\right)$.

We can choose three of the corners to form the base of the tetrahedron. Using the three corners $(0, 1, 2)$, $(3, 5, 7)$, and $(4, 6, 9)$, we can find the equation of the plane containing these three points. We use our three points and the fact that the equation of a plane is given by $z = ax + by + c$ to set up the following system of three equations and three unknowns:

$$\begin{array}{ccccccc} 2 & = & a\,(0) & + & b\,(1) & + & c \\ 7 & = & a\,(3) & + & b\,(5) & + & c \\ 9 & = & a\,(4) & + & b\,(6) & + & c \end{array}$$

Solving this system yields $a = 3$, $b = -1$, and $c = 3$. So the equation of the plane containing the base of the tetrahedron is $z = 3x - y + 3$.

We can find the height of the tetrahedron using the plane equation $3x - y - z = -3$, the fourth vertex $(8, 10, 11)$, and the following formula for the distance from a point to a plane:

$$d = \frac{|Ax + By + Cz - D|}{\sqrt{A^2 + B^2 + C^2}},$$

where (x, y, z) is the point and $Ax + By + Cz = D$ is the plane.

So the height is $h = \frac{|3(8)-1(10)-1(11)-(-3)|}{\sqrt{(3)^2+(-1)^2+(-1)^2}} = \frac{6}{\sqrt{11}}$.

To find the area of the base, recall that the base of the pyramid is the triangle formed by the points $(0,1,2)$, $(3,5,7)$, and $(4,6,9)$. The vector from $(0,1,2)$ to $(3,5,7)$ is $\langle 3,4,5\rangle$, and the vector from $(0,1,2)$ to $(4,6,9)$ is $\langle 4,5,7\rangle$. Since, geometrically, the triangle determined by the points $(0,1,2)$, $(3,5,7)$, and $(4,6,9)$ forms one half of the parallelogram defined by the two vectors, the area of the triangle is equal to one half of the magnitude of the cross product of the vectors, $|\langle 3,4,5\rangle \times \langle 4,5,7\rangle| = \sqrt{11}$. So the area of the base of the tetrahedron is $\frac{\sqrt{11}}{2}$.

As stated earlier, the volume of the tetrahedron is given by the formula $V = \frac{1}{3}Bh$. Hence, the volume is $V = \frac{1}{3}\left(\frac{6}{\sqrt{11}}\right)\left(\frac{\sqrt{11}}{2}\right) = 1$.

Problem 2 — Volume 26, No. 2, Fall, 2002

Buildings A and B are separated by a 12 foot wide alley. One 15 foot ladder rests at the base of building A and leans against the wall of building B. Another 20 foot ladder rests at the base of building B and leans against the wall of building A. What is the height where the two ladders cross?

Solution

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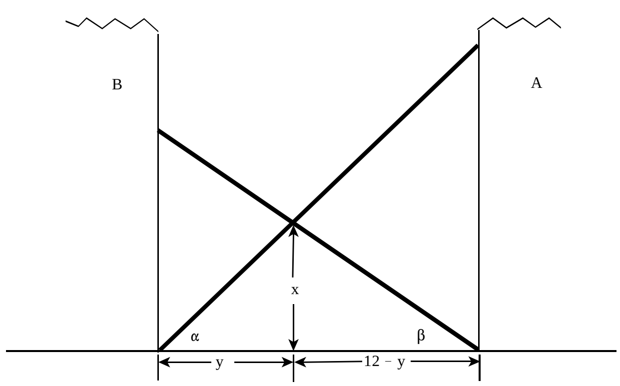

Using the Pythagorean Theorem we find that the 20 foot ladder rises to a height of $H_{20} = \sqrt{(20)^2 - (12)^2} = 16$ feet, and the 15 foot ladder rises to a height of $H_{15} = \sqrt{(15)^2 - (12)^2} = 9$ feet.

With these measurements, we know that the angle α between the ladder resting at the base of building B and the ground, has a tangent of $\frac{4}{3}$. Likewise, the angle β between the ladder resting at the base of building A and the ground, has a tangent of $\frac{3}{4}$. If we let x be the height where the two ladders cross, and y the horizontal distance from the wall of building B to the point where the two ladders cross (as shown), we have the following relationships:

$$\frac{4}{3} = \frac{x}{y} \quad \text{and} \quad \frac{3}{4} = \frac{x}{12 - y}.$$

So $4y = 3x$, $36 - 3y = 4x$, and thus $36 - \frac{9}{4}x = 4x$. Thus $x = \frac{144}{25}$.

Also solved by Eric Kleckler, Junior, Troy State University.

Problem 6 — Volume 26, No. 2, Fall, 2002.

Find three different right triangles with integer length sides such that each has an area of 840.

Solution

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With a right triangle having legs, a and b, and hypotenuse c, we know that the area is given by $A = \frac{1}{2}(ab)$. Since we want an area of 840, we know that $\frac{1}{2}(ab) = 840$, and thus ab (the product of the legs), must equal 1680. The integer 1680 has the following pairs of divisors:

$$(2, 840);\ (3, 560);\ (4, 420);\ (5, 336);\ (6, 280);\ (7, 240);\ (8, 210);$$
$$(10, 168);\ (12, 140);\ (14, 120);\ (15, 112);\ (16, 105);\ (20, 84);$$
$$(21, 80);\ (24, 70);\ (28, 60);\ (30, 56);\ (35, 48);\ \text{and}\ (40, 42).$$

The hypotenuse of the triangle, c, is the square root of the sum of the squares of the legs, $c = \sqrt{a^2 + b^2}$. In this case, c must be an integer. The only pairs of divisors giving an integer solution for c are $(15, 112)$; $(24, 70)$; and $(40, 42)$. Hence, the (only) three right triangles with area 840 and integer sides are $(15, 112, 113)$; $(24, 70, 74)$; and $(40, 42, 58)$.

Also solved by Nick Newman, Senior, and Joe Frye, Sophomore, Troy State University, Troy, AL.