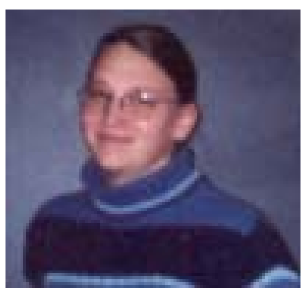
Undergraduate Research

A Generalization of the Eleven Pebble Problem

By Sheena Branton Richards

1. Biographical Sketch

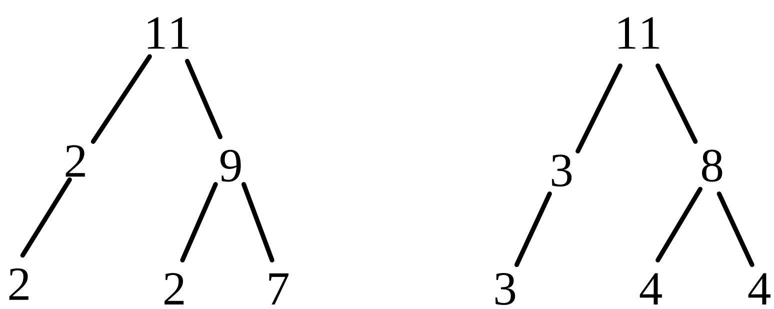


Sheena Richards, daughter of Shelburn and Karen Branton, of Taylor, Alabama, lives in Rehobeth, Alabama with her husband, Brad Richards. Currently a senior at Troy State University, she graduated valedictorian in 2000 from Rehobeth High School, where she attended grades K-12. Sheena first became interested in math beyond the classroom setting in the 8th grade, when her pre-algebra teacher, Mrs. Pat Tyson, and her Algebra 1 teacher, Ms. Denise Wilson, introduced her to MathCounts. She continued to participate in math competitions throughout high school. In addition to those teachers already mentioned, she credits "Coach" Greg Burdeshaw, Ms. Mary Ann Shields, and Ms. Kerri Pelham with having considerable influence on her mathematical development.

Editor's Note: This paper was inspired by a problem that appeared in the Spring, 2002 issue of the *Alabama Journal of Mathematics.* The problem poses an original variation on the game of *Nim.* It's statement is as follows:

> A simple game begins with 11 stones arranged in a single pile. Two players take alternating turns. Each turn consists of selecting any pile that contains at least 3 stones, and then splitting this pile into two smaller piles. The only restriction is that, after each turn, all the currently remaining piles must contain different numbers of stones. The game ends when one of the players can make no legal move, and this player is declared to be the loser. Assuming that both players want to win the game, what should be the strategy of the first player on his/her first turn?

Notice that the only restriction mentioned in the game is that no two piles of the same size may exist at the same time. There are two scenarios which result in this restriction being violated. They are shown schematically below.



Violations of the problem's restriction

Given every possible sequence of allowable moves, the game involving eleven pebbles has only two possible outcomes. Either there are four piles of size 1, 2, 3, 5 (the game ends after Player 1's second move); or there are three piles of size 1, 4, 6 (the game ends after Player 2's first move).

In particular, note that if the game ends with three piles of size 1, 4, and 6, then Player 1 loses. So, it should be the strategy of player one to prevent Player 2 from being able to produce three piles of size 1, 4, and 6 on his turn.

Working backwards, it can be shown that Player 2's first move can produce three piles of size 1, 4, and 6 only if Player 1 breaks the original pile of eleven pebbles into two piles of size 5 and 6; or two piles of size 1 and 10; or two piles of size 4 and 7. Thus, as long as Player 1 does not produce two piles of size 5 and 6; or 1 and 10; or 4 and 7 on his first turn, he/she will always win. Stated differently, as long as Player 1 chooses to break down the pile of eleven pebbles into either two piles of size 2 and 9, or two piles of size 3 and 8, he/she will automatically win.

This variation of the game of *Nim* merits further investigation. In particular, what can be said about the game if the original pile of pebbles is of arbitrary size n?

Before beginning, we formulate the following definitions:

Definition 1. *A decomposition of the original pile into smaller piles is called a* ***closed set*** *if no further decomposition is possible.*

Example 1. *In the game involving eleven pebbles, the closed sets were* $\{1, 2, 3, 5\}$ *and* $\{1, 4, 6\}$.

Definition 2. *A closed set is called an* ***even closed set*** *if it contains an even number of piles.*

Example 2. *In the game involving eleven pebbles,* $\{1, 2, 3, 5\}$ *is an even closed set.*

Definition 3. *A closed set is called an* ***odd closed set*** *if it contains an odd number of piles.*

Example 3. *In the game involving eleven pebbles,* $\{1, 4, 6\}$ *is an odd closed set.*

Definition 4. *A game beginning with a pile of* n *pebbles is a* ***fair game*** *if there exist both an even and an odd closed set, and if Player 1 cannot automatically win by making a strategic first move.*

Definition 5. *A game beginning with a pile of* n *pebbles is a* ***quasi-fair game*** *if there exists both an even and an odd closed set, and if Player 1 can automatically win by making a strategic first move.*

Example 4. *The game beginning with eleven pebbles, is a quasi-fair game. As long as Player 1 makes the "right" first move (decomposing the pile into two piles of size 2 and 9, or two piles of size 3 and 8), it is impossible for Player 2 to win.*

Definition 6. *A game beginning with a pile of* n *pebbles is a* ***fixed game*** *if there does not exist both an even closed set and an odd closed set. (i.e. A game is* ***fixed*** *if either Player 1 cannot lose or Player 1 cannot win.)*

Example 5. *Simple calculation shows that games beginning with either 3, 4, or 5 pebbles are fixed in favor of Player 1, while the game that begins with 6 pebbles is fixed in favor of Player 2.*

The reader will find it quite easy to prove the following statements about closed sets:

(1) Every closed set is accessible. That is, every closed set, the sum of whose piles equals n, can be obtained by decomposing a pile of n pebbles according to the rules and restrictions stated.
(2) Every closed set is either even or odd. In fact, the number of moves required to obtain a particular closed set is unique. If a closed set has k piles, then it is accessed through $k-1$ moves.

Having a grasp of the definitions, some less obvious questions natural arise, namely:

(1) For which values of n is the game:
    (a) fair?
    (b) quasi-fair?
    (c) fixed?
(2) How does the strategy of the players change with n?

To begin our examination of these questions, we consider all closed sets for games beginning with ten pebbles or less. (The even closed sets are in bold type.)

Number of Pebbles	Closed Sets
3	$\{\mathbf{1},\mathbf{2}\}$
4	$\{\mathbf{1},\mathbf{3}\}$
5	$\{\mathbf{1},\mathbf{4}\}$; $\{\mathbf{2},\mathbf{3}\}$
6	$\{\mathit{1,2,3}\}$
7	$\{\mathit{1,2,4}\}$
8	$\{\mathit{1,2,5}\}$; $\{\mathit{1,3,4}\}$
9	$\{\mathit{2,3,4}\}$; $\{\mathit{1,3,5}\}$; $\{\mathit{1,2,6}\}$
10	$\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$

Note that for games beginning with less than eleven pebbles, the games are fixed. Games beginning with 3, 4, 5, or 10 pebbles have no odd closed sets, and games beginning with 6, 7, 8, or 9 pebbles have no even closed sets. As we observed earlier, a game beginning with 11 pebbles is quasi-fair. Hence, the question as to which values of n produce a "fair" game now becomes more significant, for we see that no game played with than eleven pebbles or less is a fair game.

Now we continue by examining the closed sets for games beginning with more than eleven pebbles:

Number of Pebbles	Closed Sets
12	$\{\mathbf{1,2,3,6}\}\,;\{\mathbf{1,2,4,5}\}$
13	$\{\mathbf{1,2,3,7}\}\,;\{\mathbf{1,2,4,6}\}$
14	$\{\mathbf{1,2,3,8}\}\,;\{\mathbf{1,2,4,7}\}$
15	$\{\mathit{1,2,3,4,5}\}$
16	$\{\mathit{1,2,3,4,6}\}\,;\{\mathbf{1,2,5,8}\}$
17	$\{\mathit{1,2,3,4,7}\}\,;\{\mathit{1,2,3,5,6}\}$
18	$\{\mathit{1,2,3,4,8}\}\,;\{\mathit{1,2,3,5,7}\}\,;\{\mathit{1,2,4,5,6}\}\,;\ldots$
19	$\{\mathit{1,2,3,4,9}\}\,;\{\mathit{1,2,3,5,8}\}\,;\{\mathit{1,2,3,6,7}\}\,;\ldots$
20	$\{\mathit{1,2,3,4,10}\};\{\mathit{1,2,3,5,9}\};\{\mathit{1,2,3,6,8}\};\ldots$
21	$\{\mathbf{1,2,3,4,5,6}\}$
22	$\{\mathbf{1,2,3,4,5,7}\}\,;\{\mathit{1,2,3,6,10}\}$
23	$\{\mathbf{1,2,3,4,5,8}\}\,;\{\mathbf{1,2,3,4,6,7}\}$

Note that for these numbers, the only games which are not fixed are those games played with 16 or 22 pebbles. The question then arises, what significance is attached to the numbers 11, 16, and 22 which causes them to have both an even and an odd closed set?

Consider the closed sets for the numbers preceding 11, 16, and 22:

10	$\{\mathbf{1,2,3,4}\}$
15	$\{\mathit{1,2,3,4,5}\}$
21	$\{\mathbf{1,2,3,4,5,6}\}$

Each of these numbers is a triangular number. Our conjecture now becomes apparent:

"If n is the successor of a triangular number greater than 6, then the game played with n pebbles is at least quasi-fair."

The proof is rather straightforward, and hinges strongly on the observation that if a pile of size n is decomposed according to the restrictions stated in the rules, then one of the resulting piles must be of size less than $\frac{n}{2}$. Now for any successor t_k+1 of a triangular number t_k, with $k \geq 4$, consider the following two decompositions of the original pile of size t_k+1 :

$$\{1,2,3,\ldots,k-1,k+1\} \text{ and } \{1,2,3,...,k-3,k,2k-2\}\,.$$

The set $\{1,2,3,\ldots,k-1,k+1\}$ is obviously closed since it contains piles of size $1,2,3,\ldots,\lfloor\frac{k+1}{2}\rfloor$, hence no pile in the set can be decomposed. For similar reasons, none of the piles of size $1,2,3,...,k-3,k$, from second set $\{1,2,3,...,k-3,k,2k-2\}$ can be decomposed. Regarding the decomposition of the remaining pile of size $2k-2$ into two piles, the smaller of the two piles would have to be larger than $k-3$ and smaller than $\lfloor\frac{2k-2}{2}\rfloor = k-1$. (i.e., the smaller pile must be of size $k-2$.) This implies that the larger pile must be of size k. But since a pile of size k already exists in the set, such a decomposition is not possible. Hence, the set $\{1, 2, 3, ..., k-3, k, 2k-2\}$ is closed.

Finally, since the set $\{1, 2, 3, \ldots, k-1, k+1\}$ has k piles and the set $\{1, 2, 3, ..., k-3, k, 2k-2\}$ has $k-1$ piles, we are guaranteed that one of these sets will be even and the other will be odd. Hence, the game will be "at least" quasi-fair.

Having proved the conjecture, it would seem reasonable to ask the questions: "Are these the only values of n ($n = t_k + 1$) which will produce both an even and odd closed set? Are there other values of n which yield at least a quasi-fair game?"

We answer these questions by means of an algorithmic proof. The strategy is as follows: Given triangular number t_k with $k \geq 14$, produce an even and an odd closed set for every number $t_k + m$, for $0 \leq m \leq k$.

Claim 1. *Any game played with at least* 105 (t_k *with* $k = 14$) *pebbles is at least quasi-fair.*

Proof.
Step 1: For a game beginning with $t_k + m$ pebbles $(k \geq 14)$, produce a closed set of size k. (i.e., a closed set with k piles.)

Given the closed set $\{1, 2, 3, \ldots, k\}$ for t_k, our strategy is to remove k and replace it with $k + m$, for $0 \leq m \leq k$. Observe that since $m \leq k$, we have: $\frac{1}{2}(k+m) \leq k$. Hence, the possible pairs of numbers into which $(k+m)$ can be decomposed are "centered" around "medians" that are less than or equal to k, and at least one member of this pair must be less than k. But every number less than k is already in the set. Thus, $(k+m)$ cannot be broken down.

For any other element s in the set, every number less than s is already in the set, so s cannot be decomposed. Hence, the set is closed.

(Note that the same process of adding m to a certain element in the set will work, not only for k, but also for any other element in the set which is greater than $k - m$.)

Step 2: For a game beginning with $t_k + 1$ pebbles $(k \geq 14)$, produce a closed set of size $k-1$. (i.e., a closed set with $k-1$ piles.)

Given the closed set $\{1, 2, 3, \ldots, k\}$ for t_k, our strategy is to remove three elements and replace them with two elements. In order to create a closed set whose sum is one greater than t_k, we remove elements, x, y, and z, and replace them with the two elements, $\frac{x+y+z+1}{2} - 1$ and $\frac{x+y+z+1}{2} + 1$. In order for this to be possible, we impose two conditions on x, y, and z :

(1) Both new elements must be greater than k.
$\Rightarrow \frac{x+y+z+1}{2} - 1 \geq k+1$
$\Rightarrow \frac{x+y+z+1}{2}$.

(2) We choose x, y, and z to be consecutive natural numbers with x being the least even number for which the first condition holds.
$\Rightarrow \frac{x+(x+1)+(x+2)+1}{2} \geq k+2$
$\Rightarrow x \geq \frac{2k}{3}$.

For $k = 3m$, for some $m \in \mathbf{N}$, we choose $x = \frac{2k}{3}$.
For $k = 3m+1$, for some $m \in \mathbf{N}$, we choose $x = \frac{2(k+2)}{3}$.
For $k = 3m+2$, for some $m \in \mathbf{N}$, we choose $x = \frac{2(k+1)}{3}$.

Having created these new sets of size $k-1$ (i.e., $k-1$ piles) we must show that they are closed. We prove the case for $k = 3m+2$. The other two cases are similar.

In accord with our algorithm, we delete the following elements:

$$\begin{aligned} x &= \frac{2(k+1)}{3} \\ y &= \frac{2(k+1)}{3} + 1 \\ z &= \frac{2(k+1)}{2} + 2 \end{aligned}$$

and replace them with the two elements below:

$$\begin{aligned} \frac{x+y+z+1}{2} - 1 &= k+2 \\ \frac{x+y+z+1}{2} + 1 &= k+4. \end{aligned}$$

Our new set looks like this:

$$\left\{1, 2, 3, \ldots, \frac{2(k+1)}{3} - 2, \frac{2(k+1)}{3} - 1, \frac{2(k+1)}{3} + 3, \right.$$
$$\left. \frac{2(k+1)}{3} + 4, \ldots, k-1, k, k+2, k+4 \right\}.$$

It remains to show that this set is closed. Let s be an element of this set. Then the median of a pair of numbers into which s can be decomposed is no greater than $\frac{k+4}{2}$. Since $k \geq 14$ by hypothesis, we have:

$$\begin{aligned} & 4k - 3k \geq 12 + 6 - 4 \\ \Rightarrow\ & 3k + 12 \leq 4k + 4 - 6 \\ \Rightarrow\ & 3(k+4) \leq 4(k+1) - 6 \\ \Rightarrow\ & \frac{(k+4)}{2} \leq \frac{2(k+1)}{3} - 1. \end{aligned}$$

So every number less than $\frac{k+4}{2}$ is already in the set. Hence, s cannot be broken down and our set is closed.

We have just established that for a game beginning with $t_k + 1$ pebbles $(k \geq 14)$, we can create a closed set of size $k-1$ (i.e., $k-1$ piles). In order to create a closed set of size $k-1$ for every number between $t_k + 1$ and t_{k+1},we algorithmically increment the elements succeeding $\frac{2(k+1)}{3} + 2$ (the last element that we deleted from the set). In order to generate *closed* sets, we find the "capacity" of the set that we have just produced for $t_k + 1$. We begin by successively incrementing the largest element of our set as shown below:

$$\left\{1, 2, 3, \ldots, \frac{2(k+1)}{3} - 2, \frac{2(k+1)}{3} - 1, \frac{2(k+1)}{3} + 3, \frac{2(k+1)}{3} + 4, \ldots, k-1, k, k+2, k+4\right\}$$

$$\left\{1, 2, 3, \ldots, \frac{2(k+1)}{3} - 2, \frac{2(k+1)}{3} - 1, \frac{2(k+1)}{3} + 3, \frac{2(k+1)}{3} + 4, \ldots, k-1, k, k+2, k+5\right\}$$

$$\left\{1, 2, 3, \ldots, \frac{2(k+1)}{3} - 2, \frac{2(k+1)}{3} - 1, \frac{2(k+1)}{3} + 3, \frac{2(k+1)}{3} + 4, \ldots, k-1, k, k+2, k+6\right\}$$

$$\vdots$$

Since we proceed by successively incrementing the largest element, observe that the largest element of the set, obtained in this way, cannot exceed $2x$. Otherwise, the largest element can be decomposed into piles of size x and y. (i.e., the largest element cannot exceed the value of $\frac{4(k+1)}{3}$.) The amount by which largest element of the set can be incremented in this fashion is: $\frac{4(k+1)}{3} - (k+4) = \frac{k-8}{3}$, where $\frac{4(k+1)}{3}$ is the maximum value of the largest element and $(k+4)$ is the original value of the largest element.

After the largest element has been incremented to its maximum possible value, we begin incrementing the second largest element, as needed. The maximum amount by which this element (with original value $k+2$) can be incremented is given by $\left(\frac{4(k+1)}{3} - 1\right) - (k+2) = \frac{k-5}{3}$. After the second largest element of the set has been incremented to its maximum value, it still may be necessary to increment other elements of the set. If so, we proceed to increment the third largest element k of the set to its maximum value, and then the fourth largest element $k-1$, and so on, until we reach the element whose original value is $\frac{2(k+1)}{3}+3$. Each of these can be incremented by a maximum of $\frac{k-2}{3}$. Starting with the third element that can be incremented and proceeding to the last element that can be incremented, we count $k-\left(\frac{2(k+1)}{3}+2\right)=\frac{k-6}{3}$ elements, in addition to the largest two, that can be incremented.

Thus, the number of pebbles, by which the capacity of the original set can be increased by incrementing in this fashion, is given by:

$$\frac{k-8}{3}+\frac{k-5}{3}+\left(\frac{k-2}{3}\right)\left(\frac{k-6}{3}\right).$$

For any value of $k \geq 14$, we would like to be able to use the aforementioned procedure to add as many as $k-1$ pebbles to the closed set for t_k+1. This will be possible, provided that:

$$\frac{k-8}{3}+\frac{k-5}{3}+\left(\frac{k-2}{3}\right)\left(\frac{k-6}{3}\right) \geq k-1.$$

Solving this inequality yields the approximate solutions $k < -1.45$ or $k > 12.45$. Since, by hypothesis, $k \geq 14$, this inequality is true for all valid values of k.

Having used our algorithm to generate a prospective closed set with $n = t_k + m$ pebbles, it remains to show that the set is indeed closed. Let s be an element of this set. We need only address the case where $k+4 \leq s \leq \frac{4(k+1)}{3}$. The median of any pair of numbers into which s can be decomposed is at most $\frac{2(k+1)}{3}$. But every number less than $\frac{2(k+1)}{3}$ is already in the set. Therefore, the set of t_k+m pebbles, formed in this manner, is closed for all $1 \leq m \leq k$.

Step 1and Step 2 of this algorithm can be used to generate an even and an odd closed set for every "non-triangular" number greater than 105. Step 1 of our algorithm established that a game beginning with t_k pebbles has a closed set of size k, for $k \geq 2$. It can easily be shown that, for a game beginning with t_k pebbles, $k \geq 7$, the set $\{1, 2, 3, \ldots, k-3, k+1, 2k-4\}$ is a closed set of size $k-1$. (The sum of the piles is t_k and if s is any element in the set, the set contains every element less than $\lfloor \frac{s}{2} \rfloor$.)

Thus, every game beginning with at least 105 pebbles has both an even and an odd closed set. □

Corollary 1. *There are only finitely many fixed games.*

This generalization of our original version of Nim begs further investigation. Among the questions awaiting exploration are:

(1) What upper and lower bounds can be placed on the size of a closed set in a game beginning with n pebbles.
(2) Given a natural number m, what is the smallest number of pebbles with which a game can begin and yield a closed set having m as its largest element?
(3) Although every game beginning with $n \geq 105$ pebbles is at least quasi-fair, what can be said about the distribution of even and odd closed sets as $n \to \infty$?

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